ROMANIAN MATHEMATICAL MAGAZINE

$Let\ k_a, k_b, k_c\ be\ the\ symmedian\ cevians\ of\ \Delta ABC. Prove\ that:$

$$\sum[k_a(b^2+c^2)] \geq 216r^3$$

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$Lemma\ 1:$

$$k_a = \frac{2bc}{b^2+c^2}m_a$$

$Proof:$

$Let\ the\ symmedian\ of\ vertex\ A\ divide\ the\ side\ \ BC\ into\ two\ parts, m\ and\ n, at\ point\ D.$

$$\frac{b^2}{m} = \frac{c^2}{n}.\quad BD = b^2t, CD = c^2t.$$

$$b^2t + c^2t = a \Rightarrow t = \frac{a}{b^2+c^2}$$

$According\ to\ Stewart's\ theorem:$

$$k_a^2 = \frac{b^2c^2t + c^2b^2t}{a} - b^2c^2t = \frac{2b^2c^2t}{b^2t+c^2t} - b^2c^2t = \frac{2b^2c^2}{b^2+c^2} - b^2c^2.\left(\frac{a}{b^2+c^2}\right) =$$

$$= \frac{b^2c^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2} = \frac{4b^2c^2}{(b^2+c^2)^2}.m_a^2 \Rightarrow k_a = \frac{2bc}{b^2+c^2}.m_a$$

$$LHS = 2\sum bcm_a \geq 6\sqrt[3]{(abc)^2m_am_bm_c} \geq 6\sqrt[3]{16r^2s^2R^2s^2r} = 6r\sqrt[3]{16s^4R^2} \geq$$

$$\geq 6r\sqrt[3]{16.729r^44r^2} = 6r.4.9r^2 = 216r^3$$

$$Equality\ holds\ for\ a = b = c.$$