ROMANIAN MATHEMATICAL MAGAZINE

$\text{İn } \Delta ABC, I_a, I_b, I_c - excenters. Prove\ that:$

$$[I_aBC]+[I_bAC]+[I_cAB]\geq 3F$$

Proposed by Sarkhan Adgozalov-Georgia

Solution 1 by Qurban Muellim-Azerbaijan

$Let\ a=x+y, b=y+z, c=x+z, s=x+y+z$

$Lemma:$

$$(x+y)(y+z)(x+z)\geq 8xyz$$

$$[I_aBC]=\frac{a.r_a}{2}$$

$$LHS=\sum\left(\frac{r_a.a}{2}\right)=\frac{1}{2}\sum\left(\frac{aF}{s-a}\right)=\frac{F}{2}\sum\frac{1}{s-a}=\frac{F}{2}\sum\frac{x+y}{z}\geq$$

$$\geq\frac{3F}{2}.\sqrt[3]{\frac{(x+y)(y+z)(x+z)}{xyz}}\geq\frac{3F}{2}\sqrt[3]{\frac{8xyz}{xyz}}=\frac{3F}{2}.2=3F$$

$Equality\ holds\ for\ a=b=c.$

Solution 2 by Ertan Yildirim-Turkiye

$Lemma:$

$$r_ar_br_c=p^2r$$

$Proof:$

$$r_a=\frac{F}{p-a}, r_b=\frac{F}{p-b}, r_c=\frac{F}{p-c}$$

$$r_ar_br_c=\frac{F^3}{(p-a)(p-b)(p-c)}=\frac{F^3p}{p(p-a)(p-b)(p-c)}=\frac{F^3p}{F^2}=Fp=prp=p^2r.$$

$$[I_aBC]=\frac{ar_a}{2}$$

$$[I_bAC]=\frac{br_b}{2}$$

$$[I_cAB]=\frac{cr_c}{2}$$

$$LHS=\frac{1}{2}\sum ar_a\geq\frac{1}{2}.3.\sqrt[3]{abcr_ar_br_c}=\frac{3}{2}.\sqrt[3]{4prRp^2r}=\frac{3}{2}.\sqrt[3]{4p^3Rr^2}\geq\frac{3}{2}.2pr=3F$$

$Equality\ holds\ for\ a=b=c.$