ROMANIAN MATHEMATICAL MAGAZINE

İn $\Delta ABC, I$ and O incenter and circumcenter.
$\angle IAO = \alpha, \angle IBO = \beta, \angle IOC = \phi$. Prove that:

$$\cos(\alpha).\sin\left(\frac{A}{2}\right) + \cos(\beta).\sin\left(\frac{B}{2}\right) + \cos(\phi).\sin\left(\frac{C}{2}\right) = 1 + \frac{r}{R}$$

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Lemma 1:

$$AI = \frac{r}{\sin\left(\frac{A}{2}\right)}$$

Lemma 2:

$$OI^2 = R(R - 2r)\ (Euler)$$

Lemma 3:

$$\sum \cos A = 1 + \frac{r}{R}, 1 - \cos A = 2\sin^2\left(\frac{A}{2}\right)$$

$$\text{Let } \Delta AIO \Rightarrow \cos(\alpha) = \frac{AI^2 + R^2 - IO^2}{2R.AI}$$

$$\text{Let } \Delta BIO \Rightarrow \cos(\beta) = \frac{BI^2 + R^2 - IO^2}{2R.BI}$$

$$\text{Let } \Delta CIO \Rightarrow \cos(\phi) = \frac{CI^2 + R^2 - IO^2}{2R.CI}$$

$$LHS = \sum \cos(\alpha).\sin\left(\frac{A}{2}\right) = \sum \frac{AI^2 + R^2 - IO^2}{2R.AI}.\sin\left(\frac{A}{2}\right) = \sum \frac{\frac{r^2}{\sin^2\left(\frac{A}{2}\right)} + R^2 - R^2 + 2Rr}{2R.\frac{r}{\sin\left(\frac{A}{2}\right)}} =$$

$$= \sum \frac{r^2 + 2Rr.\sin^2\left(\frac{A}{2}\right)}{2Rr} = \frac{3r^2 + Rr\sum 2\sin^2\left(\frac{A}{2}\right)}{2Rr} = \frac{3r^2 + Rr.(3 - \sum \cos A)}{2Rr} =$$

$$= \frac{3r^2 + Rr\left(3 - 1 - \frac{r}{R}\right)}{2Rr} = \frac{(3r^2 + 2Rr - r^2)}{2Rr} = 1 + \frac{r}{R}$$