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If in $\Delta ABC, I-incenter, O_A, O_B, O_C$ circumcenters of $\Delta BIC, \Delta AIC, \Delta AIB$ then:

$$\frac{s}{2r^2}\ge\frac{b+c}{AO_A.AI}+\frac{c+a}{BO_B.BI}+\frac{a+b}{CO_C.CI}\ge\frac{3\sqrt{3}}{R}$$

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Lemma 1:

$$AO_A=AI+R_A\ (Let\ R_A\ circumradii\ of\ \Delta BIC)$$

$$Proof: 180-\frac{A+B}{2}+90-\frac{C}{2}=270-\frac{A+B+C}{2}=180\Rightarrow A-I-O_A\ are\ collinear$$

*Lemma*2:

$$R_A=\frac{a}{2\cos\left(\frac{A}{2}\right)}$$

$$Proof:\frac{a}{\sin\left(180-\frac{B+C}{2}\right)}=2\mathrm{R_A}\ \Rightarrow \mathrm{R_A}=\frac{\mathrm{a}}{2\cos\left(\frac{A}{2}\right)}$$

Lemma 3:

$$\cos^2\left(\frac{\mathrm{A}}{2}\right)=\frac{s(s-a)}{bc}$$

$$\sum\frac{b+c}{AO_A.AI}=\sum\frac{b+c}{(AI+R_A)AI}=\sum\frac{b+c}{\frac{r^2}{\sin\left(\frac{A}{2}\right)}+\frac{ar}{2\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)}}=\sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right)}{r^2+\frac{ar}{2}.\tan\left(\frac{A}{2}\right)}=$$

$$=\sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right)}{r^2+\frac{ar^2}{2}.\frac{1}{s-a}}=\sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right)}{r^2\left(1+\frac{a}{2(s-a)}\right)}=\sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right).2(s-a)}{r^2(2s-2a+a)}=$$

$$=\sum\frac{2\mathrm{s(s-a)}\sin^2\left(\frac{A}{2}\right)}{sr^2}=\sum\frac{2bc\cos^2\left(\frac{A}{2}\right)\sin^2\left(\frac{A}{2}\right)}{sr^2}=$$

$$=\sum\frac{bc\left(2\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)\right)^2}{2sr^2}=\sum\frac{bc\sin^2(A)}{2sr^2}=$$

$$=\sum\frac{bca^2}{8sR^2r^2}=\frac{abc}{8sR^2r^2}.\sum a=\frac{s}{rR}\ge\frac{3\sqrt{3}}{R}.$$

$$Again,\qquad \frac{s}{rR}\le\frac{s}{2r^2}.$$