ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC, I-incenter. O_A, O_B, O_C$ circumcenters of $\Delta BIC, \Delta AIC, \Delta AIB$.
Prove that:

$$[AO_BC]+[AO_CB]+[BO_AC]\ge 3\sqrt{3}r^2$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by proposer

$$Let\ \angle BO_AC=180-A, \Delta BO_AC\Rightarrow 2R_A=\frac{a}{\sin\left(90+\frac{A}{2}\right)}\Rightarrow R_A=\frac{a}{2\cos\left(\frac{A}{2}\right)}.$$

$$[BO_AC]=\frac{1}{2}.R_A^2.\sin(180-A)=\frac{1}{2}.\frac{a^2}{4\cos\left(\frac{A}{2}\right)^2}.sinA=\frac{a^2}{4}.\tan\left(\frac{A}{2}\right)$$

$$LHS=\sum\frac{a^2}{4}.\tan\left(\frac{A}{2}\right)=\frac{1}{4}.\sum a^2.\tan\left(\frac{A}{2}\right)=\frac{1}{4}\sum a^2.\left(\frac{r_a}{s}\right)=\frac{1}{4s}.\sum\left(\frac{a^2}{\frac{1}{r_a}}\right)\ge$$

$$\ge\frac{1}{4s}.\frac{4s^2}{\frac{1}{r}}=sr\ge 3\sqrt{3}r^2$$

Equality holds for $a=b=c$.