ROMANIAN MATHEMATICAL MAGAZINE

If I_a, I_b, I_c – excenters then in ΔABC the following relationship holds:

$$\frac{AI_a}{\sin\frac{A}{2}}+\frac{BI_b}{\sin\frac{B}{2}}+\frac{CI_c}{\sin\frac{C}{2}}\geq 36r$$

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Solution by Ertan Yildirim-Turkiye

Lemma: $AI_a=\frac{s}{\cos\frac{A}{2}}$

Proof: We know that the points A, I and I_a are collinear. Therefore, $\angle I_aAC=\frac{A}{2}$

Let F be the point where the excircle opposite A in tangent to AC. Then

$\cos\frac{A}{2}=\frac{AF}{AI_a}$. Since: $AF=b+s-b=s$, it follows that $AI_a=\frac{s}{\cos\frac{A}{2}}$

$$LHS=\sum\frac{AI_a}{\sin\left(\frac{A}{2}\right)}=\sum\left(\frac{2s}{\sin A}\right)=2s\sum\frac{2R}{a}=4Rs\sum\left(\frac{1}{a}\right)\geq 4Rs.\left(\frac{\sqrt{3}}{R}\right)=$$

$$\geq 4\sqrt{3}s\geq 4\sqrt{3}.3\sqrt{3}r=36r.$$

Equality holds for $a=b=c$.