ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC holds :

$$\frac{1}{bc(r_a-r)}+\frac{1}{ac(r_b-r)}+\frac{1}{ab(r_c-r)}\le\left(\frac{1}{2r}\right)^3$$

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Lemma 1. $S=pr=(p-a)r_a=(p-b)r_b=(p-c)r_c,\qquad abc=4RF$

Lemma 2. $r_a-r=\dfrac{ar}{p-a},\qquad r_a=\dfrac{S}{p-a},\qquad r=\dfrac{F}{p},r_a-r=\dfrac{aF}{p(p-a)}$

Lemma 3. $R\overset{Euler}{\geq}2r\ \xrightarrow{Lemma\ 2}\begin{cases}\dfrac{1}{bc(r_a-r)}=\dfrac{p-a}{abcr}\\[2mm]\dfrac{1}{ac(r_b-r)}=\dfrac{p-b}{abcr}\\[2mm]\dfrac{1}{ab(r_c-r)}=\dfrac{p-c}{abcr}\end{cases}$

$$LHS=\sum_{cyc}\frac{1}{bc(r_a-r)}=\frac{3p-(a+b+c)}{abcr}=\frac{3p-2p}{abcr}=\frac{p}{abcr}$$

$$LHS\overset{Lemma\ 1;2}{\cong}\frac{\frac{S}{r}}{4RSr}=\frac{1}{4Rr^2}\quad R\overset{Euler}{\geq}2r\ \ \frac{1}{R}\le\frac{1}{2r}\to\frac{1}{4Rr^2}\le\frac{1}{8r^3}$$

$$\sum_{cyc}\frac{1}{bc(r_a-r)}\le\left(\frac{1}{2r}\right)^3$$