

ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC : I_a, I_b, I_c excenters of ΔABC

(H_A, R_a) (H_B, R_b) (H_C, R_c) orthocenters and circumradius of $(\Delta I_a BC)$, $(\Delta I_b AC)$ respectively $(\Delta I_c AB)$. Prove that:

$$\frac{I_a H_A}{R_a} + \frac{I_b H_B}{R_b} + \frac{I_c H_C}{R_c} \leq 3$$

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Lemma 1: In ΔABC with orthocenter H and circumradius R

$$AH = 2R|\cos(A)|$$

Lemma 2: $\angle BI_a C = 90^\circ - \frac{A}{2}$, $\angle I_a BC = 90^\circ - \frac{B}{2}$, $\angle I_a CB = 90^\circ - \frac{C}{2}$

Lemma 3: $f(x) = \sin\left(\frac{x}{2}\right)$ $x \in (0; \pi) \rightarrow \sin\left(\frac{x}{2}\right) > 0$

$$\frac{d^2}{dx^2} f(x) = -\frac{1}{4} \sin\left(\frac{x}{2}\right) \rightarrow f''(x) < 0 \text{ concave function on } (0; \pi)$$

$$\frac{\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)}{3} \stackrel{JENSEN}{\leq} \sin\left(\frac{A+B+C}{6}\right) = \sin\left(\frac{\pi}{6}\right)$$

$$\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) \leq \frac{3}{2}$$

$$I_a H_A = 2R_a |\cos(\angle BI_a C)| \stackrel{Lemma 2}{=} 2R_a \cos\left(90^\circ - \frac{A}{2}\right) = 2R_a \sin\left(\frac{A}{2}\right)$$

$$\frac{I_a H_A}{R_a} = 2 \sin\left(\frac{A}{2}\right) \rightarrow \sum_{cyc} \frac{I_a H_A}{R_a} = 2 \underbrace{\sum_{cyc} \sin\left(\frac{A}{2}\right)}_{\stackrel{Lemma 3}{\leq} 2 \cdot \frac{3}{2}} = 3$$

$$\frac{I_a H_A}{R_a} + \frac{I_b H_B}{R_b} + \frac{I_c H_C}{R_c} \leq 3$$