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$$Let\ P\ be\ the\ Brocard's\ point\ of\ acute-angled\ ABC.$$

$$R_a, R_b, R_c\ circumradius\ of\ BPC, APC, APB.$$

$$H-orthocenter. Prove\ that:$$

$$\frac{R_a \bullet h_a}{AH} + \frac{R_b \bullet h_b}{BH} + \frac{R_c \bullet h_c}{CH} \ge 9r$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$Lemma\ 1: cosA \bullet cosB \bullet cosC \le \frac{1}{8}$

$Lemma\ 2: h_a = 2R \bullet sinB \bullet sinC$

$$\frac{a}{\sin(180-C)} = 2R_a => R_a = \frac{a}{2sinC}$$

$$\text{LHS} = \sum \frac{R_a \bullet h_a}{AH} = \sum \frac{\frac{a}{2sinC} \bullet 2RsinBsinC}{2RcosA} = \sum \frac{asinB}{2cosA} \ge \frac{3}{2}\sqrt[3]{\frac{abc\prod sinA}{\prod cosA}}$$

$$\ge 3\sqrt[3]{abc\prod sinA} = 3\sqrt[3]{abc\left(\frac{abc}{8R^3}\right)} = 3\sqrt[3]{\frac{(abc)^2}{8R^3}} = 3\sqrt[3]{\frac{16R^2r^2s^2}{8R^3}}$$

$$= 3\sqrt[3]{\frac{2s^2r^2}{R}} \ge 3\sqrt[3]{\left(\frac{27Rrr^2}{R}\right)} = 9r$$

Equality holds for a=b=c.