ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{b+c}{m_a}}+\sum_{cyc}\sqrt{\frac{m_a}{b+c}}+\frac{R^5}{32r^5}\ge 1+\sum_{cyc}\sqrt{\frac{b+c}{n_a}}+\sum_{cyc}\sqrt{\frac{n_a}{b+c}}$$

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$$n_a^2\overset{\text{Bogdan Fustei}}{=}s^2-2r_ah_a\therefore a^2n_a^2\overset{?}{\le}4(R-r)^2s^2$$

$$\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\le}4(R-r)^2s^2$$

$$\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\le}4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\ge}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\ge}0\to\text{true}\therefore n_a\le\frac{2(R-r)s}{a}\text{ and analogs}\Rightarrow$$

$$\sum_{cyc}\sqrt{\frac{n_a}{b+c}}\le\sum_{cyc}\sqrt{\frac{2(R-r)s}{a(b+c)}}=\sum_{cyc}\sqrt{\frac{2(R-r)bcs}{4Rrs(b+c)}}=\frac{R-r}{\sqrt{4Rr(R-r)}}.\sum_{cyc}\sqrt{\frac{2bc}{b+c}}$$

$$\overset{\substack{\text{Euler}\\\text{and}\\\text{AM-HM}}}{\le}\frac{R-r}{\sqrt{8Rr^2}}.\sum_{cyc}\sqrt{b+c}\overset{\text{CBS}}{\le}\frac{R-r}{\sqrt{8Rr^2}}.\sqrt{3.4s}\overset{\text{Mitrinovic}}{\le}\frac{R-r}{\sqrt{2Rr^2}}.\sqrt{3R.\frac{3\sqrt{3}}{2}}$$

$$=\frac{R-r}{2r}.3.\sqrt[4]{3}\therefore\sum_{cyc}\sqrt{\frac{n_a}{b+c}}\overset{①}{\le}\left(t^2-\frac{1}{2}\right).3.\sqrt[4]{3}\left(t=\sqrt{\frac{R}{2r}}\right)\text{ and again,}$$

$$\sum_{cyc}\sqrt{\frac{m_a}{b+c}}\overset{\text{Lascu}}{\ge}\frac{1}{\sqrt{2}}.\sum_{cyc}\sqrt{\cos\frac{A}{2}}\overset{\text{AM-GM}}{\ge}\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{s}{4R}}\overset{\text{Mitrinovic}}{\ge}\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{3\sqrt{3}.r}{4R}}$$

$$=\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{3\sqrt{3}.rR^2}{4R^3}}\overset{\text{Euler}}{\ge}\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{3\sqrt{3}.4r^3}{4R^3}}=\frac{3.\sqrt[4]{3}}{2}.\sqrt{\frac{2r}{R}}\therefore\sum_{cyc}\sqrt{\frac{m_a}{b+c}}\overset{②}{\ge}\frac{3.\sqrt[4]{3}}{2}.\frac{1}{t}\text{ and}$$

$$\because n_a\overset{\text{Bogdan Fustei}}{\ge}m_a\text{ and analogs}\therefore\sum_{cyc}\sqrt{\frac{b+c}{m_a}}\overset{③}{\ge}\sum_{cyc}\sqrt{\frac{b+c}{n_a}}\text{ and so,}$$

$$①,②\text{ and }③\Rightarrow\text{it suffices to prove : }\frac{3.\sqrt[4]{3}}{2}.\frac{1}{t}+t^{10}\overset{?}{\ge}1+\left(t^2-\frac{1}{2}\right).3.\sqrt[4]{3}$$

$$\Leftrightarrow t^{10}-1\overset{?}{\ge}\frac{3.\sqrt[4]{3}}{2}.\left(2t^2-1-\frac{1}{t}\right)\Leftrightarrow$$

$$(t-1)\left(t^{10}+t^{9}+t^{8}+t^{7}+t^{6}+t^{5}+t^{4}+t^{3}+t^{2}+t\right)\overset{?}{\geq}\frac{3.\sqrt[4]{3}}{2}.(t-1)\begin{pmatrix}2t^{2}+\\2t+1\end{pmatrix}$$

$$and \because t-1\overset{Euler}{\geq}0 \text{ and } \frac{3.\sqrt[4]{3}}{2}<2 \therefore \text{it suffices to prove :}$$

$$t^{10}+t^{9}+t^{8}+t^{7}+t^{6}+t^{5}+t^{4}+t^{3}+t^{2}+t\overset{?}{\geq}4t^{2}+4t+2 \rightarrow \text{true}$$

$$\because t^{10}+t^{9}+t^{8}+t^{7}\overset{t\geq 1}{\geq}t^{2}+t^{2}+t^{2}+t^{2}=4t^{2} \text{ and}$$

$$t^{6}+t^{5}+t^{4}+t^{3}\overset{t\geq 1}{\geq}t+t+t+t=4t \text{ and finally}, t^{2}+t\overset{t\geq 1}{\geq}1+1=2$$

$$\therefore \sum_{cyc}\sqrt{\frac{b+c}{m_a}}+\sum_{cyc}\sqrt{\frac{m_a}{b+c}}+\frac{R^5}{32r^5}\geq 1+\sum_{cyc}\sqrt{\frac{b+c}{n_a}}+\sum_{cyc}\sqrt{\frac{n_a}{b+c}} \ \forall \Delta ABC,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$