ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}+\frac{1013R^3}{4r^3}\geq 2026+\sum\sqrt{\frac{m_a+m_b}{m_c}}$$

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$$\sum\sqrt{\frac{w_a+w_b}{w_c}}\overset{AM-GM}{\geq}\sum\frac{\sqrt{2}\sqrt[4]{w_aw_b}}{\sqrt{w_c}}\overset{AM-GM}{\geq}3\sqrt{2}$$

$$\frac{(m_a+m_b)}{m_c}=\frac{m_a}{m_c}+\frac{m_b}{m_c}\overset{Panaitopol\ \&\ m_c\geq h_c}{\leq}\frac{\frac{Rh_a}{2r}}{h_c}+\frac{\frac{Rh_b}{2r}}{h_c}=\frac{R}{2r}\left(\frac{h_a}{h_c}+\frac{h_b}{h_c}\right)=\frac{R}{2r}\left(\frac{c}{a}+\frac{c}{b}\right)$$

$$\sum\sqrt{\frac{m_a+m_b}{m_c}}\leq\sqrt{\frac{R}{2r}}\sum\sqrt{\left(\frac{c}{a}+\frac{c}{b}\right)}\overset{CBS}{\leq}\sqrt{\frac{R}{2r}}\sqrt{3\sum\left(\frac{c}{a}+\frac{c}{b}\right)}=$$

$$=\sqrt{\frac{R}{2r}}\sqrt{3\sum\left(\frac{c}{a}+\frac{a}{c}\right)}\overset{Bandila}{\leq}\sqrt{\frac{R}{2r}}\sqrt{\frac{9R}{r}}=\frac{3\sqrt{2}R}{r}$$

We need to show:

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}+\frac{1013R^3}{4r^3}\geq 2026+\sum\sqrt{\frac{m_a+m_b}{m_c}}$$

$$3\sqrt{2}+\frac{1013R^3}{4r^3}\geq 2026+\frac{3\sqrt{2}R}{r}$$

$$1013x^3-6\sqrt{2}x+12\sqrt{2}-8104\overset{\frac{R}{r}=x\geq 2\ Euler}{\geq}0$$

$$(x-2)\left(1013(x^2+2x+4)-6\sqrt{2}\right)\geq 0\ true\ as\ x\geq 2$$

Equality holds for an equilateral triangle.