ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\frac{1}{IA}+\frac{1}{IB}+\frac{1}{IC}\le\frac{1}{HA}+\frac{1}{HB}+\frac{1}{HC}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{HA}=\frac{1}{4R\cos A\cos B\cos C}.2\sum_{cyc}(\cos B\cos C)=$$

$$=\frac{4R^2}{4R(s^2-(2R+r)^2)}.\left(\left(\frac{R+r}{R}\right)^2-\left(3-\frac{2(s^2-4Rr-r^2)}{4R^2}\right)\right)$$

$$=\frac{4R^2}{4R(s^2-(2R+r)^2)}.\frac{s^2-4R^2+r^2}{2R^2}=\frac{s^2-4R^2+r^2}{2R(s^2-(2R+r)^2)}\overset{?}{\ge}\frac{3}{2r}$$

$$\Leftrightarrow 12R^3+8R^2r+3Rr^2+r^3\underset{(*)}{\overset{?}{\ge}}(3R-r)s^2$$

$$\text{Now, }(3R-r)s^2\overset{\text{Blundon-Gerretsen}}{\le}(3R-r).\frac{R(4R+r)^2}{4R-2r}\overset{?}{\le}12R^3+8R^2r+3Rr^2+r^3$$

$$\Leftrightarrow r^2(R^2-Rr-2r^2)\overset{?}{\ge}0\Leftrightarrow r^2(R-2r)(R+r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{1}{HA}\ge\frac{3}{2r}\overset{\text{Jensen}}{\ge}\frac{1}{r}\sum_{cyc}\sin\frac{A}{2}=\sum_{cyc}\frac{1}{IA}\text{ and so,}$$

$$\frac{1}{IA}+\frac{1}{IB}+\frac{1}{IC}\le\frac{1}{HA}+\frac{1}{HB}+\frac{1}{HC}\ \forall\text{ acute }\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$