ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1+\cot A}{\sin^2 A}+\frac{1+\cot B}{\sin^2 B}+\frac{1+\cot C}{\sin^2 C}\ge\frac{12+4\sqrt{3}}{3}$$

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Solution by Tapas Das-India

$$Let\ x=\cot A, y=\cot B, z=\cot C\ \ then\ \sum xy=\sum cotA.cotB=1\ (1)$$

$$\sum x\ge\sqrt{3(xy+yz+zx)}\overset{(1)}{\cong}\sqrt{3}\ (2)$$

$$\frac{1+\cot A}{\sin^2 A}+\frac{1+\cot B}{\sin^2 B}+\frac{1+\cot C}{\sin^2 C}=\sum\frac{1+\cot A}{\sin^2 A}=\sum(1+cotA)\csc^2 A=$$

$$=\sum(1+x)(1+x^2)=\sum(1+x+x^2+x^3)=$$

$$=3+\sum x+\sum x^2+\sum x^3\overset{CBS}{\ge}3+\sum x+\frac{1}{3}\left(\sum x\right)^2+\frac{1}{9}\left(\sum x\right)^3$$

$$\overset{(2)}{\ge}3+\sqrt{3}+\frac{3}{3}+\frac{1}{9}\left(\sqrt{3}\right)^3=4+\sqrt{3}+\frac{\sqrt{3}}{3}=\frac{12+4\sqrt{3}}{3}$$

Equality holds for an equilateral triangle.