ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{w_a}{b+c}+\frac{w_b}{a+c}+\frac{w_c}{a+b}\leq\frac{3\sqrt{3}}{4}$$

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$$w_a=\frac{2\cdot b\cdot c\cdot\cos\left(\frac{A}{2}\right)}{b+c}\rightarrow\frac{w}{b+c}=\frac{2\cdot b\cdot c\cdot\cos\left(\frac{A}{2}\right)}{(b+c)^2}$$

$$\frac{b+c}{2}\overset{AM-GM}{\geqq}\sqrt{bc}\rightarrow\frac{4}{(b+c)^2}\leq\frac{1}{bc}\rightarrow\frac{bc}{(b+c)^2}\leq\frac{1}{4}$$

$$\frac{w_a}{b+c}\leq\frac{1}{2}\cos\left(\frac{A}{2}\right)\rightarrow\sum_{cyc}\frac{w_a}{b+c}\leq\frac{1}{2}\sum_{cyc}\cos\left(\frac{A}{2}\right)$$

$$RHS=\frac{1}{2}\sum_{cyc}\cos\left(\frac{A}{2}\right)\rightarrow f(x)=\cos\left(\frac{x}{2}\right)\rightarrow\frac{d^2}{dx^2}f(x)=-\frac{1}{4}\cos\left(\frac{x}{2}\right)<0$$

$f(x)$ is concave function

$$\frac{\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)}{3}\overset{JENSEN}{\leqq}\cos\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)\rightarrow\angle A+\angle B+\angle C=\pi$$

$$\sum_{cyc}\cos\left(\frac{A}{2}\right)\leq 3\cos\left(\frac{\pi}{6}\right)=\frac{3\sqrt{3}}{2}\rightarrow RHS=\frac{1}{2}\sum_{cyc}\cos\left(\frac{A}{2}\right)\leq\frac{3\sqrt{3}}{4}$$

$$\sum_{cyc}\frac{I_a}{b+c}\leq\frac{1}{2}\sum_{cyc}\cos\left(\frac{A}{2}\right)\rightarrow\sum_{cyc}\frac{I_a}{b+c}\leq\frac{3\sqrt{3}}{4}$$

The equality holds for an equilateral triangle.