ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\frac{m_a^2+m_b^2}{\cos C}+\frac{m_b^2+m_c^2}{\cos A}+\frac{m_c^2+m_a^2}{\cos B}\geq 27R^2$$

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$$\text{We have } \forall\, m,n,p,u,v,w>0,\ (n+p)u+(p+m)v+(m+n)w\geq 2\sqrt{(mn+np+pm)(uv+vw+wu)}\to (1)$$

$$\text{So, }\sum_{cyc}\frac{m_b^2+m_c^2}{\cos A}=\sum_{cyc}\left(m_a^2\left(\frac{1}{\cos B}+\frac{1}{\cos C}\right)\right)=(n+p)u+(p+m)v+(m+n)w$$

$$\left(m=\frac{1}{\cos A},n=\frac{1}{\cos B},p=\frac{1}{\cos C}\ \text{and}\ u=m_a^2,v=m_b^2,w=m_c^2\right)\overset{\text{via }(1)}{\geq}$$

$$2.\sqrt{\sum_{cyc}m_a^2m_b^2}.\sqrt{\sum_{cyc}\frac{1}{\cos B\cos C}}=2.\sqrt{\frac{9}{16}\left(\sum_{cyc}a^2b^2\right).\frac{R+r}{R\left(\frac{s^2-4R^2-4Rr-r^2}{4R^2}\right)}}\overset{?}{\geq}27R^2$$

$$\Leftrightarrow (R+r)((s^2+4Rr+r^2)^2-16Rrs^2)\overset{?}{\geq}81R^3(s^2-4R^2-4Rr-r^2)$$

$$\Leftrightarrow (R+r)s^4-(81R^3+8R^2r+6Rr^2-2r^3)s^2+$$

$$324R^5+324R^4r+97R^3r^2+24R^2r^3+9Rr^4+r^5\ \overset{?}{\underset{(*)}{\geq}}\ 0$$

$$\text{Now, since } P=(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore \text{ in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*)\overset{?}{\geq}P$$

$$\Leftrightarrow 324R^5+324R^4r-159R^3r^2-72R^2r^3+144Rr^4-24r^5\ \overset{?}{\underset{(**)}{\geq}}$$

$$(81R^3-24R^2r-16Rr^2+8r^3)s^2$$

$$\text{Again, }(81R^3-24R^2r-16Rr^2+8r^3)s^2\overset{\text{Gerretsen}}{\leq}$$

$$(81R^3-24R^2r-16Rr^2+8r^3)(4R^2+4Rr+3r^2)\overset{?}{\leq}\text{LHS of }(**)$$

$$\Leftrightarrow 48t^4-121t^3+16t^2+80t-24\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(48t^2+71t+108)+228\right)\overset{?}{\geq}0\to \text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\Rightarrow(*)$$

$$\text{is true}\ \therefore \frac{m_a^2+m_b^2}{\cos C}+\frac{m_b^2+m_c^2}{\cos A}+\frac{m_c^2+m_a^2}{\cos B}\geq 27R^2\ \forall \text{ acute } \Delta\, ABC,$$

$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$