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In any Δ ABC the following relationship holds :

$$\frac{a}{l_b+l_c}+\frac{b}{l_c+l_a}+\frac{c}{l_a+l_b}\ge\sqrt{3}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{l_b+l_c}=\sum_{cyc}\frac{a^2}{a(l_b+l_c)}\overset{\text{Bergstrom}}{\ge}\frac{4s^2}{\sum_{cyc}\left(a(l_b+l_c)\right)}=\frac{4s^2}{\sum_{cyc}\left((b+c)l_a\right)}$$

$$=\frac{4s^2}{\sum_{cyc}\left((b+c)\frac{2bc}{b+c}\cos\frac{A}{2}\right)}=\frac{2s^2}{\sum_{cyc}\left(bc.\sqrt{\frac{s(s-a)}{bc}}\right)}=\frac{2s^2}{\sum_{cyc}\left(\sqrt{bc}.\sqrt{s(s-a)}\right)}$$

$$\overset{\text{CBS}}{\ge}\frac{2s^2}{\sqrt{\sum_{cyc}ab}\,.\sqrt{s\sum_{cyc}(s-a)}}\ge\frac{2s^2}{\sqrt{\frac{4s^2}{3}}.\sqrt{s^2}}=\sqrt{3}\text{ and so,}$$

$$\frac{a}{l_b+l_c}+\frac{b}{l_c+l_a}+\frac{c}{l_a+l_b}\ge\sqrt{3}\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$