ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\le\frac{b^2c^2}{16r^4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s_0 = \text{semiperimeter and then : } \cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\overset{?}{\le}\frac{b^2c^2}{16r^4}$$

$$\Leftrightarrow \frac{s_0}{r}.\frac{s_0(s_0-a)}{r.s_0}\overset{?}{\le}\frac{16R^2r^2s_0^2}{16r^4.a^2}\Leftrightarrow a^2(s_0-a)\overset{?}{\le}R^2s_0$$

$$\Leftrightarrow 16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\overset{?}{\le}R^2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\Leftrightarrow 16s^2(1-s^2)(c-s)\overset{?}{\le}s+c\ \left(s=\sin\frac{A}{2}, c=\cos\frac{B-C}{2}\right)$$

$$\Leftrightarrow (16s^2(1-s^2)-1)c\ \boxed{\underset{(*)}{\overset{?}{\le}}}\ s+16s^3(1-s^2)$$

Case 1 $16s^2(1-s^2)>1$ and then, since $c\le 1\ \therefore$ LHS of $(*)\le 16s^2(1-s^2)-1$

$$\overset{?}{\le}s+16s^3(1-s^2)\Leftrightarrow 16s^2(1-s)\overset{?}{\le}1+16s^3(1-s)\Leftrightarrow 16s^2(1-s)^2\overset{?}{\le}1$$

$$\Leftrightarrow 4s(1-s)\overset{?}{\le}1\Leftrightarrow 4s^2-4s+1\overset{?}{\ge}0\Leftrightarrow(1-2s)^2\overset{?}{\ge}0\to\text{true}\Rightarrow(*)\text{ is true}$$

Case 2 $16s^2(1-s^2)\le 1$ and then, since $c=\cos\frac{B-C}{2}=\frac{b+c}{a}.\sin\frac{A}{2}>\sin\frac{A}{2}=s$

$$\therefore \text{LHS of }(*)\le(16s^2(1-s^2)-1)s\overset{?}{\le}s+16s^3(1-s^2)\Leftrightarrow 2s\overset{?}{\ge}0\to\text{true}$$

(strict inequality) $\Rightarrow (*)$ is true $\therefore$ combining both cases, $(*)$ is true $\forall\,\Delta$ ABC

$$\therefore \cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\le\frac{b^2c^2}{16r^4}\ \forall\,\Delta\text{ ABC}, ''='' \text{ iff } \cos\frac{B-C}{2}=1 \text{ and } \sin\frac{A}{2}=\frac{1}{2}$$

$\Rightarrow$ "=" iff Δ ABC is equilateral (QED)