ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\frac{\cos A}{1+\tan^2 A}+\frac{\cos B}{1+\tan^2 B}+\frac{\cos C}{1+\tan^2 C}\geq\frac{3}{8}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cos A}{1+\tan^2 A}=\sum_{cyc}\cos^3 A=$$

$$=3\prod_{cyc}\cos A+\frac{1}{2}\left(\sum_{cyc}\cos A\right)\left(3\sum_{cyc}\cos^2 A-\left(\sum_{cyc}\cos A\right)^2\right)$$

$$=\frac{3(s^2-(2R+r)^2)}{4R^2}+\frac{R+r}{2R}.\left(3\left(3-\frac{s^2-4Rr-r^2}{2R^2}\right)-\frac{(R+r)^2}{R^2}\right)$$

$$=\frac{3(s^2-(2R+r)^2)-2(R+r)^3+18R^2(R+r)-3(R+r)(s^2-4Rr-r^2)}{4R^3}\overset{?}{\geq}\frac{3}{8}$$

$$\Leftrightarrow 5R^3+24R^2r+12Rr^2+2r^3\underset{(*)}{\overset{?}{\geq}}6rs^2$$

$$\text{Now, }6rs^2\overset{\text{Gerretsen}}{\leq}6r(4R^2+4Rr+3r^2)\overset{?}{\leq}R^3+24R^2r+12Rr^2+2r^3$$

$$\Leftrightarrow 5R^3-12Rr^2-16r^3\overset{?}{\geq}0\Leftrightarrow(R-2r)(5R^2+10Rr+8r^2)\overset{?}{\geq}0\rightarrow\text{true}$$

$$\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(*)\text{ is true}\therefore\frac{\cos A}{1+\tan^2 A}+\frac{\cos B}{1+\tan^2 B}+\frac{\cos C}{1+\tan^2 C}\geq\frac{3}{8}$$

$\forall\,\Delta$ ABC (QED), $''=''$ iff Δ ABC is equilateral (QED)