ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sqrt{\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}}}+\sqrt{\frac{\tan\frac{C}{2}\tan\frac{A}{2}}{\tan\frac{B}{2}}}\geq\sqrt[4]{27}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$We\ know\ that\ in\ any\ triangle\sum\tan\frac{A}{2}\tan\frac{B}{2}=1(1)$$

$$let\tan\frac{A}{2}=x,\tan\frac{B}{2}=y,\tan\frac{C}{2}=z\ then\ xy+yz+zx\overset{(1)}{=}1\ (2)$$

$$\sqrt{\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}}}+\sqrt{\frac{\tan\frac{C}{2}\tan\frac{A}{2}}{\tan\frac{B}{2}}}=$$

$$=\sqrt{\frac{xy}{z}}+\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}=\frac{xy+yz+zx}{\sqrt{xyz}}\overset{(2)}{=}\frac{1}{\sqrt{xyz}}=\frac{1}{\sqrt[4]{x^2y^2z^2}}=$$

$$=\sqrt[4]{\frac{1}{xy.yz.zx}}\overset{AM-GM}{\geq}\sqrt[4]{\frac{27}{(xy+yz+zx)^3}}\overset{(2)}{=}\sqrt[4]{27}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{\sqrt{3}}\Rightarrow A=B=C=\frac{\pi}{3}$$