ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{a}{\sqrt{b^2+c^2}}+\frac{b}{\sqrt{c^2+a^2}}+\frac{c}{\sqrt{a^2+b^2}}<2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{\sqrt{b^2+c^2}}\overset{\text{Reverse CBS}}{\leq}\sqrt{2}.\sum_{cyc}\frac{a}{b+c}=\sqrt{2}.\sum_{cyc}\frac{a}{s+s-a}<\sqrt{2}.\sum_{cyc}\frac{a}{s}$$

$$=\sqrt{2}.\frac{2s}{s}=2\sqrt{2}\ \forall\ \Delta\ ABC$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$Any\ \Delta ABC\rightarrow\begin{cases}a+b>c\\ b+c>a\\ a+c>b\end{cases}\rightarrow\begin{cases}\dfrac{c}{a+b}<1\\ \dfrac{a}{b+c}<1\\ \dfrac{b}{a+c}<1\end{cases}$$

$$a<b+c\rightarrow\ a+b+c<2(b+c)$$

$$\rightarrow\frac{1}{b+c}<\frac{2}{a+b+c}\rightarrow\frac{a}{b+c}<\frac{2a}{a+b+c}$$

$$\sum_{cyc}\frac{a}{b+c}<\sum_{cyc}\frac{2a}{a+b+c}\rightarrow\sum_{cyc}\frac{a}{b+c}<2$$

$$(c-b)^2\geq 0\rightarrow\ c^2+b^2\geq 2cb\rightarrow\ c^2+b^2\geq\frac{(c+b)^2}{2}$$

$$\frac{c+b}{\sqrt{2}}\leq\sqrt{c^2+b^2}\rightarrow\frac{1}{\sqrt{c^2+b^2}}\leq\frac{\sqrt{2}}{c+b}\rightarrow\frac{a}{\sqrt{c^2+b^2}}\leq\frac{a\sqrt{2}}{b+c}$$

$$\sum_{cyc}\frac{a}{\sqrt{b^2+c^2}}\leq\sqrt{2}\sum_{cyc}\frac{a}{b+c}$$

$$\sum_{cyc}\frac{a}{b+c}<2\rightarrow\sum_{cyc}\frac{a}{\sqrt{b^2+c^2}}<2\sqrt{2}$$