ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\big((\sin A-\sin B)(\cot B+\cot C)\big)\ge 0$$

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$$\sum_{cyc}\big((\sin A-\sin B)(\cot B+\cot C)\big)=$$

$$=\sum_{cyc}\left((\sin A-\sin B)\left(\sum_{cyc}\cot A-\cot A\right)\right)$$

$$=\left(\sum_{cyc}\cot A\right)\left(\sum_{cyc}(\sin A-\sin B)\right)+\sum_{cyc}\frac{a(b^2+c^2-a^2)(b-a)}{2abc.\,2R}$$

$$=\frac{1}{4Rabc}.\sum_{cyc}\big(a(b^2+c^2-a^2)(b-a)\big)\text{ and so, it remains to prove :}$$

$$\sum_{cyc}\big(a(b^2+c^2-a^2)(b-a)\big)\underbrace{\overset{?}{\ge}}_{(*)}0$$

Now, we assign : $s-a\equiv x, s-b\equiv y, s-c\equiv z$ and then : $(*)\Leftrightarrow$

$$\sum_{cyc}\big((y+z)(x-y)((z+x)^2+(x+y)^2-(y+z)^2)\big)\overset{?}{\ge}0$$

$$\Leftrightarrow\sum_{cyc}x^3y+\sum_{cyc}x^2y^2\overset{?}{\ge}2xyz\sum_{cyc}x\to\text{true}\because\sum_{cyc}x^3y=xyz.\sum_{cyc}\frac{x^2}{z}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{xyz\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x}=xyz\sum_{cyc}x\text{ and }\sum_{cyc}x^2y^2\ge xyz\sum_{cyc}x\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\big((\sin A-\sin B)(\cot B+\cot C)\big)\ge 0\ \forall\ \Delta\ ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$