ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{a-b}{a^2+bc}+\frac{b-c}{b^2+ca}+\frac{c-a}{c^2+ab}\le 0$$

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$$\frac{a-b}{a^2+bc}+\frac{b-c}{b^2+ca}+\frac{c-a}{c^2+ab}$$

$$=\frac{1}{(a^2+bc)(b^2+ca)(c^2+ab)}.\sum_{cyc}\left((b-c)(c^2a^2+bc^3+a^3b+abc.b)\right)$$

$$=\frac{1}{(a^2+bc)(b^2+ca)(c^2+ab)}.\left(\begin{array}{c} abc\sum_{cyc}ab+\sum_{cyc}b^2c^3+\sum_{cyc}a^3b^2+abc\sum_{cyc}a^2- \\ \sum_{cyc}a^3b^2-\sum_{cyc}bc^4-abc\sum_{cyc}a^2-abc\sum_{cyc}ab\end{array}\right)$$

$$=\frac{1}{(a^2+bc)(b^2+ca)(c^2+ab)}.\left(\sum_{cyc}b^2c^3-\sum_{cyc}bc^4\right)\text{ and so, it remains}$$

to prove : $\sum_{cyc}bc^4 \underset{(*)}{\overset{?}{\ge}} \sum_{cyc}b^2c^3$ & now, we assign : $s-a\equiv x, s-b\equiv y, s-c\equiv z$

$$\text{and then : } (*) \Leftrightarrow \sum_{cyc}\left((z+x)(x+y)^4\right)\overset{?}{\ge}\sum_{cyc}\left((z+x)^2(x+y)^3\right)$$

$$\Leftrightarrow \sum_{cyc}x^4y+\sum_{cyc}x^3y^2+\sum_{cyc}x^2y^3 \underset{(**)}{\overset{?}{\ge}} 3xyz\sum_{cyc}xy$$

$$\text{Now, }\sum_{cyc}x^4y+\sum_{cyc}x^2y^3+\sum_{cyc}x^3y^2 \overset{AM-GM}{\ge} 2\sum_{cyc}x^3y^2+\sum_{cyc}x^3y^2=3\sum_{cyc}\frac{x^3y^3}{y}$$

$$\overset{Holder}{\ge} 3.\frac{\left(\sum_{cyc}xy\right)^3}{3\sum_{cyc}x}\ge\frac{\left(\sum_{cyc}xy\right).3xyz\sum_{cyc}x}{\sum_{cyc}x}=3xyz\sum_{cyc}xy\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{a-b}{a^2+bc}+\frac{b-c}{b^2+ca}+\frac{c-a}{c^2+ab}$$

$$\le 0\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\text{ ABC is equilateral (QED)}$$