

# ROMANIAN MATHEMATICAL MAGAZINE

In any  $\Delta ABC$  the following relationship holds :

$$\frac{r_a - h_a}{b + c} + \frac{r_b - h_b}{c + a} + \frac{r_c - h_c}{a + b} \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \sum_{\text{cyc}} \frac{r_a - h_a}{b + c} &= \frac{1}{(b + c)(c + a)(a + b)} \cdot \sum_{\text{cyc}} \left( (r_a - h_a) \left( a^2 + \sum_{\text{cyc}} ab \right) \right) = \\ &= \frac{1}{(b + c)(c + a)(a + b)} \cdot \left( \left( \sum_{\text{cyc}} ab \right) \left( 4R + r - \sum_{\text{cyc}} h_a \right) + \sum_{\text{cyc}} a^2 r_a - \sum_{\text{cyc}} \left( a^2 \cdot \frac{2rs}{a} \right) \right) \\ &\stackrel{\text{Chebyshev}}{\geq} \frac{1}{(b + c)(c + a)(a + b)} \cdot \left( \left( \sum_{\text{cyc}} ab \right) \left( 4R + r - \sum_{\text{cyc}} m_a \right) + \right. \\ &\quad \left. \frac{1}{3} \left( \sum_{\text{cyc}} a^2 \right) (4R + r) - 4s^2 r \right) \\ &(\because \text{WLOG assuming } a \geq b \geq c \Rightarrow a^2 \geq b^2 \geq c^2 \text{ and } r_a \geq r_b \geq r_c) \stackrel{\text{Bager and Euler}}{\geq} \\ &= \frac{1}{(b + c)(c + a)(a + b)} \cdot \left( \left( \sum_{\text{cyc}} ab \right) \left( \sum_{\text{cyc}} m_a - \sum_{\text{cyc}} m_a \right) + \right. \\ &\quad \left. \frac{2}{3} (s^2 - 4Rr - r^2)(9r) - 4s^2 r \right) \\ &= \frac{1}{(b + c)(c + a)(a + b)} \cdot 2r(s^2 - 12Rr - 3r^2) \\ &= \frac{1}{(b + c)(c + a)(a + b)} \cdot 2r(s^2 - 16Rr + 5r^2 + 4r(R - 2r)) \geq 0 \\ &\quad \left( \because s^2 - 16Rr + 5r^2 \stackrel{\text{Gerretsen}}{\geq} 0 \text{ and } R - 2r \stackrel{\text{Euler}}{\geq} 0 \right) \\ \therefore \frac{r_a - h_a}{b + c} + \frac{r_b - h_b}{c + a} + \frac{r_c - h_c}{a + b} &\geq 0 \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$