ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{h_a - h_b}{b + c} + \frac{h_b - h_c}{c + a} + \frac{h_c - h_a}{a + b} \leq 0$$

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$$\sum_{cyc} \frac{h_a - h_b}{b + c} = \frac{2rs}{(b + c)(c + a)(a + b).4Rrs} \cdot \sum_{cyc} \left(c(b - a) \left(a^2 + \sum_{cyc} ab \right) \right)$$

$$= \frac{1}{(b + c)(c + a)(a + b).2R} \cdot \left(\left(\sum_{cyc} ab \right) \left(c(b - a) + a(c - b) + b(a - c) \right) + abc \sum_{cyc} a - \sum_{cyc} ca^3 \right)$$

$$= \frac{1}{(b + c)(c + a)(a + b).2R} \cdot \left(abc \sum_{cyc} a - abc. \sum_{cyc} \frac{a^2}{b} \right) \overset{\text{Bergstrom}}{\leq}$$

$$\frac{1}{(b + c)(c + a)(a + b).2R} \cdot \left(abc \sum_{cyc} a - abc. \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a} \right) = 0$$

$$\therefore \frac{h_a - h_b}{b + c} + \frac{h_b - h_c}{c + a} + \frac{h_c - h_a}{a + b} \leq 0 \;\; \forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$