ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{b}{r_a}+\frac{c}{r_b}+\frac{a}{r_c}\geq\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}$$

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Solution by Daniel Sitaru-Romania

$$\frac{b}{r_a}+\frac{c}{r_b}+\frac{a}{r_c}\geq\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}\Leftrightarrow\sum_{cyc}\frac{b}{r_a}\geq\sum_{cyc}\frac{b}{h_a}\Leftrightarrow$$

$$\sum_{cyc}\frac{b}{\frac{F}{s-a}}\geq\sum_{cyc}\frac{b}{\frac{2F}{a}}\Leftrightarrow 2\sum_{cyc}b(s-a)\geq\sum_{cyc}ab$$

$$2s\sum_{cyc}b-2\sum_{cyc}ab\geq\sum_{cyc}ab\Leftrightarrow 4s^2\geq 3\sum_{cyc}ab$$

$$4s^2\geq 3(s^2+r^2+4Rr)\Leftrightarrow s^2\geq 3r^2+12Rr \text{ (to prove)}$$

$$s^2\overset{GERRETSEN}{\geq}16Rr-5r^2\geq 3r^2+12Rr\Leftrightarrow 4Rr\geq 8r^2\Leftrightarrow R\geq 2r\text{(Euler)}$$

Equality holds for $a=b=c$.