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In any ΔABC the following relationship holds :

$$\sum_{\text{cyc}} \frac{\sec \frac{A}{2}}{1 + \tan \frac{A}{2}} \geq \frac{6}{1 + \sqrt{3}}$$

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$$\begin{aligned} \text{Let } f(x) &= \frac{\sec \frac{x}{2}}{1 + \tan \frac{x}{2}} \quad \forall x \in (0, \pi) \text{ and then : } f''(x) = \\ &= \frac{\sec^3 \frac{x}{2} + \sec \frac{x}{2} \tan \frac{x}{2} (\tan^2 \frac{x}{2} + \tan \frac{x}{2} - \sec^2 \frac{x}{2})}{4 \left(1 + \tan \frac{x}{2}\right)^2} - \frac{\sec^3 \frac{x}{2} (\tan^2 \frac{x}{2} + \tan \frac{x}{2} - \sec^2 \frac{x}{2})}{2 \left(1 + \tan \frac{x}{2}\right)^3} \\ &= \frac{3 - \sin x}{4 \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^3} \geq 0 \quad \left(\because \sin x \leq 1 < 3 \text{ and } \sin \frac{x}{2}, \cos \frac{x}{2} > 0 \text{ as } \frac{x}{2} \in \left(0, \frac{\pi}{2}\right) \right) \\ \therefore f(x) &= \frac{\sec \frac{x}{2}}{1 + \tan \frac{x}{2}} \text{ is convex } \forall x \in (0, \pi) \text{ and so, } \sum_{\text{cyc}} \frac{\sec \frac{A}{2}}{1 + \tan \frac{A}{2}} \stackrel{\text{Jensen}}{\geq} 3 \cdot \frac{\sec \frac{\pi}{6}}{1 + \tan \frac{\pi}{6}} \\ &= \frac{6}{1 + \sqrt{3}} \quad \forall \Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$