ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\sum_{cyc}\frac{\sec\frac{A}{2}}{1+\tan\frac{A}{2}}\geq\frac{6}{1+\sqrt{3}}$$

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$$\text{Let } f(x)=\frac{\sec\frac{x}{2}}{1+\tan\frac{x}{2}}\ \forall\in(0,\pi)\text{ and then : } f''(x)=$$

$$\frac{\sec^3\frac{x}{2}+\sec\frac{x}{2}\tan\frac{x}{2}\left(\tan^2\frac{x}{2}+\tan\frac{x}{2}-\sec^2\frac{x}{2}\right)}{4\left(1+\tan\frac{x}{2}\right)^2}-\frac{\sec^3\frac{x}{2}\left(\tan^2\frac{x}{2}+\tan\frac{x}{2}-\sec^2\frac{x}{2}\right)}{2\left(1+\tan\frac{x}{2}\right)^3}$$

$$=\frac{3-\sin x}{4\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^3}\geq 0\ \left(\because \sin x\leq 1<3 \text{ and } \sin\frac{x}{2},\cos\frac{x}{2}>0\text{ as }\frac{x}{2}\in\left(0,\frac{\pi}{2}\right)\right)$$

$$\therefore f(x)=\frac{\sec\frac{x}{2}}{1+\tan\frac{x}{2}}\text{ is convex}x\ \forall\in(0,\pi)\text{ and so, }\sum_{cyc}\frac{\sec\frac{A}{2}}{1+\tan\frac{A}{2}}\overset{\text{Jensen}}{\geq}3.\frac{\sec\frac{\pi}{6}}{1+\tan\frac{\pi}{6}}$$

$$=\frac{6}{1+\sqrt{3}}\ \forall\,\Delta\,ABC,''=''\ \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$