ROMANIAN MATHEMATICAL MAGAZINE

In acute $\Delta ABC$ the following relationship holds:

$$\sum_{cyc} \csc\left(\frac{A}{2}\right)\left(1+\tan\left(\frac{A}{2}\right)\right) \geq 6+2\sqrt{3}$$

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$$\sum_{cyc} \csc\left(\frac{A}{2}\right)\left(1+\tan\left(\frac{A}{2}\right)\right) = \sum_{cyc}\left(\frac{1}{\sin\left(\frac{A}{2}\right)}+\frac{1}{\cos\left(\frac{A}{2}\right)}\right)$$

$$Let\ \ f(x) = \frac{1}{\sin(x)}+\frac{1}{\cos(x)},\quad \frac{A}{2}=x,\ \ x\in\left(0;\frac{\pi}{2}\right)$$

$$f'(x) = \left(\frac{1}{\sin(x)}+\frac{1}{\cos(x)}\right)' = -\frac{\cos(x)}{sin^2(x)}+\frac{\sin(x)}{cos^2(x)},$$

$$f''(x) = \frac{1+cos^2(x)}{sin^3(x)}+\frac{1+sin^2(x)}{cos^3(x)} > 0$$

$$f(x)\ \ is\ \ convexe\ function.\ By\ \ Jensen's\ \ inequality:$$

$$\sum_{cyc}\left(\frac{1}{\sin\left(\frac{A}{2}\right)}+\frac{1}{\cos\left(\frac{A}{2}\right)}\right) \geq 3.\frac{1}{\sin\left(\frac{A+B+C}{6}\right)+\cos\left(\frac{A+B+C}{6}\right)} =$$

$$3\left(\frac{1}{\sin\left(\frac{\pi}{6}\right)}+\frac{1}{\cos\left(\frac{\pi}{6}\right)}\right) = 3\left(2+\frac{2}{\sqrt{3}}\right) = 6+2\sqrt{3}$$

$$Equality\ \ holds\ \ for\ an\ equilateral\ \ triangle.$$