ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sec(A)}{1+\tan(A)}\ge\frac{6}{1+\sqrt{3}}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\sec(A)}{1+\tan(A)}=\sum_{cyc}\frac{\frac{1}{\cos(A)}}{1+\tan(A)}=\sum_{cyc}\frac{1}{\sin(A)+\cos(A)}\to$$

$$A\in\left(0;\frac{\pi}{2}\right)\quad=\frac{1}{\sqrt{2}}\sum_{cyc}\frac{1}{\sin\left(\frac{\pi}{4}+A\right)}$$

$$Let\ \ f(x)=\frac{1}{\sin(x)},\qquad x\in\left(\frac{\pi}{4};\frac{3\pi}{4}\right),\qquad f'(x)=-\frac{\cos(x)}{sin^2(x)},$$

$$f''(x)=\frac{1+cos^2(x)}{sin^3(x)}>0\ \ So,\qquad f(x)\ \ is\ \ convexe\ function$$

$$By\ \ Jensen's\ \ inequality:$$

$$\sum_{cyc}\frac{1}{\sin(A)+\cos(A)}\ge 3.\frac{1}{\sin\left(\frac{A+B+C}{3}\right)+\cos\left(\frac{A+B+C}{3}\right)}=\frac{3}{\frac{\sqrt{3}}{2}+\frac{1}{2}}=\frac{6}{1+\sqrt{3}}$$

$$Equality\ \ holds\ for\ an\ equilateral\ \ triangle.$$