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In ΔABC the following relationship holds:

$$\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}+\frac{\csc\left(\frac{B}{2}\right)}{1+\cos\left(\frac{B}{2}\right)}+\frac{\csc\left(\frac{C}{2}\right)}{1+\cos\left(\frac{C}{2}\right)}\geq\frac{12}{2+\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}\geq\frac{12}{2+\sqrt{3}}\rightarrow LHS=\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}=\sum_{cyc}\frac{1}{\sin\left(\frac{A}{2}\right)\left(1+\cos\left(\frac{A}{2}\right)\right)}$$

$$A+B+C=\pi\rightarrow\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\quad x\in\left(0;\frac{\pi}{2}\right)\rightarrow\begin{cases}0<\sin(x)<1\\0<\cos(x)<1\end{cases}$$

$$f(x)=\frac{1}{\sin(x)(1+\cos(x))}\qquad\frac{d^2}{dx}f(x)>0\rightarrow Convexity$$

$$Jensen's\ inequality:\ \frac{\sum_{i=1}^{n}f(x_i)}{n}\geq f\left(\sum_{i=1}^{n}x_i\right)$$

$$\frac{1}{3}\sum_{cyc}\frac{1}{\sin\left(\frac{A}{2}\right)\left(1+\cos\left(\frac{A}{2}\right)\right)}\geq f\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=f\left(\frac{\pi}{6}\right)$$

$$\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}\geq\frac{3}{\sin\left(\frac{\pi}{6}\right)\left(1+\cos\left(\frac{\pi}{6}\right)\right)}=\frac{3}{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)}$$

$$\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}\geq\frac{12}{2+\sqrt{3}}$$

Equality holds for an equilateral triangle.