ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$2\left(\frac{1}{\cos A}+\frac{1}{\cos B}\right)\ge\sqrt{3}(\tan A+\tan B)+2$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\Delta\ \text{ABC being acute}\Rightarrow\cos A,\cos B>0$$

$$\therefore 2\left(\frac{1}{\cos A}+\frac{1}{\cos B}\right)\overset{?}{\ge}\sqrt{3}.(\tan A+\tan B)+2$$

$$\Leftrightarrow 2(\cos A+\cos B)\overset{?}{\ge}\sqrt{3}.(\sin A\cos B+\cos A\sin B)+2\cos A\cos B$$

$$=\sqrt{3}.\sin C+\cos(A+B)+\cos(A-B)$$

$$\Leftrightarrow 4\sin\frac{C}{2}\cos\frac{A-B}{2}\overset{?}{\ge}\sqrt{3}.\sin C-\cos C+2\cos^2\frac{A-B}{2}-1$$

$$\Leftrightarrow 2t^2-4\sin\frac{C}{2}.t+\sqrt{3}.\sin C-2\cos^2\frac{C}{2}\underset{(*)}{\overset{?}{\le}}0\ \left(t=\cos\frac{A-B}{2}\right)\ \text{and to prove }(*),$$

$$\text{it suffices to prove : } t\underset{①}{\overset{?}{\le}}\frac{4\sin\frac{C}{2}+\sqrt{16\sin^2\frac{C}{2}-8\left(\sqrt{3}.\sin C-2\cos^2\frac{C}{2}\right)}}{4}\ \text{AND}$$

$$t\underset{②}{\overset{?}{\ge}}\frac{4\sin\frac{C}{2}-\sqrt{16\sin^2\frac{C}{2}-8\left(\sqrt{3}.\sin C-2\cos^2\frac{C}{2}\right)}}{4};\because t\le 1\ \therefore\ \text{in order to prove ①,}$$

$$\text{it suffices to prove : } 1\overset{?}{\le}\sin\frac{C}{2}+\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}\Leftrightarrow\left(1-\sin\frac{C}{2}\right)^2\overset{?}{\le}1-\frac{\sqrt{3}}{2}.\sin C$$

$$\Leftrightarrow\sin^2\frac{C}{2}-2\sin\frac{C}{2}+\sqrt{3}.\sin\frac{C}{2}\cos\frac{C}{2}\overset{?}{\le}0\Leftrightarrow\sin\frac{C}{2}+\sqrt{3}.\cos\frac{C}{2}\overset{?}{\le}2\to\text{true}$$

$$\because\sin\frac{C}{2}+\sqrt{3}.\cos\frac{C}{2}\overset{CBS}{\le}\sqrt{1+3}.\sqrt{\sin^2\frac{C}{2}+\cos^2\frac{C}{2}}=2\Rightarrow ①\ \text{is true with equality iff}$$

$$\frac{C}{2}=\frac{\pi}{6}\wedge A=B\Rightarrow\text{equality iff } A=B=C=\frac{\pi}{3}\ \text{and again},0<A,B<\frac{\pi}{2}\Rightarrow$$

$$-\frac{\pi}{4}<\frac{A-B}{2}<\frac{\pi}{4}\Rightarrow t\ge\frac{1}{\sqrt{2}}\overset{?}{>}\sin\frac{C}{2}-\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}$$

$$\Leftrightarrow\frac{1}{2}+1-\frac{\sqrt{3}}{2}.\sin C+2.\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}\overset{?}{>}\sin^2\frac{C}{2}\ \text{and}\because\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}>0$$

$$\therefore\ \text{it suffices to prove : } 3-\sqrt{3}.\sin C\overset{?}{>}1-\cos C\Leftrightarrow\cos\frac{\pi}{6}\sin C-\sin\frac{\pi}{6}\cos C\overset{?}{<}1$$

$$\Leftrightarrow\sin\left(C-\frac{\pi}{6}\right)\overset{?}{<}1\to\text{true}\Rightarrow ②\ \text{is true (strict inequality) and so, ① and ②}$$

$$\boldsymbol{are}\ \text{true} \Rightarrow (*)\ \text{is true} \therefore 2\left(\frac{1}{\cos A}+\frac{1}{\cos B}\right) \ge \sqrt{3}.(\tan A+\tan B)+2$$

$$\forall\ \boldsymbol{acute}\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral}\ (QED)$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$f(x)=\frac{1}{\cos(x)}-\sqrt{3}\tan(x) \to f(A)+f(B)\ge 2$$

$$\textit{Investigation of extreme values}:$$

$$f(x)=2\sec(x)-\sqrt{3}\tan(x) \to \frac{d}{dx}f(x)=2\sec(x)\tan(x)-\sqrt{3}\sec^2(x)$$

$$\frac{d}{dx}f(x)=\frac{2\sin(x)-\sqrt{3}}{\cos^2(x)}\quad \frac{d}{dx}f(x)=0 \to \sin(x)=\frac{\sqrt{3}}{2}$$

$$\textit{The unique root in the acute interval}$$

$$\left(0;\frac{\pi}{2}\right)\ \textit{is}\ x=\frac{\pi}{3}$$

$$\textit{Characterization of the function and global minimum}$$

- $x\in\left(0;\frac{\pi}{3}\right) \to \sin(x)<\frac{\sqrt{3}}{2}\quad \frac{d}{dx}f(x)<0\ \ \textit{decreasing function}$
- $x\in\left(\frac{\pi}{3};\frac{\pi}{2}\right) \to \sin(x)>\frac{\sqrt{3}}{2}\quad \frac{d}{dx}f(x)>0\ \ \textit{increasing function}$

$$\textit{The minimum point of the function}\ x=\frac{\pi}{3}$$

$$f\left(\frac{\pi}{3}\right)=\frac{1}{\cos\left(\frac{\pi}{3}\right)}-\sqrt{3}\tan\left(\frac{\pi}{3}\right)=1$$

$$x\in\left(0;\frac{\pi}{2}\right) \to\ f(x)\ge 1\ \to \begin{cases} f(A)\ge 1\\ f(B)\ge 1\end{cases}\ f(A)+f(B)\ge 2$$

$$2\left(\frac{1}{\cos(A)}+\frac{1}{\cos(B)}\right)\ge\sqrt{3}(\tan(A)+\tan(B))+2\quad (Q.E.D)$$