ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\frac{\cos A\cos B}{\cos C}+\frac{\cos B\cos C}{\cos A}+\frac{\cos C\cos A}{\cos B}+4\cos A\cos B\cos C\ge 2$$

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$$\sum_{cyc}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sum_{cyc}\frac{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{r}{R}$$

$$=\frac{r^2}{16R^2}\cdot\frac{4R}{r}\cdot\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}+\frac{r}{R}=\frac{1}{4Rr}\sum_{cyc}AI^2+\frac{r}{R}=\frac{1}{4Rr}\left(\sum_{cyc}bc-12Rr\right)+\frac{r}{R}$$

$$=\frac{s^2-8Rr+r^2+4r^2}{4Rr}\overset{\text{Gerretsen}}{\ge}\frac{8Rr}{4Rr}=2\ \therefore\ \sum_{cyc}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ge 2$$

and implementing it on a triangle with angles : $(\pi-2A),(\pi-2B),(\pi-2C)$,

we get : $$\sum_{cyc}\frac{\sin\frac{\pi-2B}{2}\sin\frac{\pi-2C}{2}}{\sin\frac{\pi-2A}{2}}+4\sin\frac{\pi-2A}{2}\sin\frac{\pi-2B}{2}\sin\frac{\pi-2C}{2}\ge 2$$

$\Rightarrow \sum_{cyc}\frac{\cos B\cos C}{\cos A}+4\cos A\cos B\cos C\ge 2$ and since the latter triangle is an

acute one, hence : $\frac{\cos A\cos B}{\cos C}+\frac{\cos B\cos C}{\cos A}+\frac{\cos C\cos A}{\cos B}+4\cos A\cos B\cos C\ge 2$

$\forall$ acute Δ ABC, " = " iff Δ ABC is equilateral (QED)