ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$(2-\cos^2 A)(2-\cos^2 B)(2-\cos^2 C) > 4$$

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$$cos2A + cos2B + cos2C = 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1 =$$

$$\overset{A+B+C=\pi}{=} -2cosCcos(A-B) + 2\cos^2 C - 1 = -1 + 2cosC(cosC - \cos(A-B)) =$$

$$\overset{A+B+C=\pi}{=} 2cosC(-\cos(A+B) - \cos(A-B)) - 1 = -1 - 4cosAcosBcosC$$

$$\sin^2 A + \sin^2 B + \sin^2 C = \sum \sin^2 A =$$

$$= \frac{1}{2}\sum 2\sin^2 A = \frac{1}{2}\sum(1-\cos 2A) = \frac{3-(cos2A+cos2B+cos2C)}{2} =$$
$$= \frac{1}{2}(3+1+4cosAcosBcosC) = 2 + 2cosAcosBcosC >$$

$$> 2 \ (\textit{in acute } cosAcosBcosC > 0)$$

$$Let\ \sin^2 A = x, \sin^2 B = y, \sin^2 C = z \ and \ x+y+z > 2$$

$$as\ x^2 = \sin^4 A < \sin^2 A = x \ as \ 0 < \sin A < 1 \text{ for this } x^2+y^2+z^2 < x+y+z \ (1)$$

$$(2-\cos^2 A)(2-\cos^2 B)(2-\cos^2 C) = (1+\sin^2 A)(1+\sin^2 B)(1+\sin^2 C) =$$

$$= (1+x)(1+y)(1+z) = 1 + (x+y+z) + (xy+yz+zx) + xyz >$$

$$> 1 + (x+y+z) + (xy+yz+zx) = 1 + (x+y+z) + \frac{(x+y+z)^2 - (x^2+y^2+z^2)}{2}$$

$$\overset{(1)}{>} 1 + (x+y+z) + \frac{(x+y+z)^2 - (x+y+z)}{2} = \frac{2 + (x+y+z) + (x+y+z)^2}{2} >$$

$$> \frac{2+2+2^2}{2} = 4 \ as\ x+y+z > 2$$