ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$7\sum a^4+64s^2Rr\geq 5(s^2+r^2+4Rr)^2$$

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$$ab+bc+ca=s^2+r^2+4Rr,\qquad abc=4Rrs$$

$$7\sum a^4+64s^2Rr\geq 5(s^2+r^2+4Rr)^2$$

$$7\sum a^4+8.(2s).(4Rrs)\geq 5(s^2+r^2+4Rr)^2$$

$$7\sum a^4+8abc(a+b+c)\geq 5(ab+bc+ca)^2\ \ (1)$$

$$7\sum a^4+8abc(a+b+c)=$$

$$=7\left((a^2+b^2+c^2)^2-2(a^2b^2+b^2c^2+c^2a^2)\right)+8abc(a+b+c)=$$

$$=7(a^2+b^2+c^2)^2-14(a^2b^2+b^2c^2+c^2a^2)+8abc(a+b+c)\geq$$

$$\underset{\substack{x^2+y^2+z^2\geq xy+yz+zx\\ (x+y+z)^2\geq 3(xy+yz+zx)}}{\overset{\forall x,y,z>0}{\geq}} 7\times 3(a^2b^2+b^2c^2+c^2a^2)-14(a^2b^2+b^2c^2+c^2a^2)$$

$$+8abc(a+b+c)=7(a^2b^2+b^2c^2+c^2a^2)+8abc(a+b+c)$$

$$5(ab+bc+ca)^2=5(a^2b^2+b^2c^2+c^2a^2)+10abc(a+b+c)$$

$$Now\ from(1)we\ need\ to\ show:$$

$$7(a^2b^2+b^2c^2+c^2a^2)+8abc(a+b+c)\geq$$

$$\geq 5(a^2b^2+b^2c^2+c^2a^2)+10abc(a+b+c)$$

$$2(a^2b^2+b^2c^2+c^2a^2)\geq 2abc(a+b+c)$$

$$this\ is\ true\ as\ 2(a^2b^2+b^2c^2+c^2a^2)\geq 2(ab.bc+bc.ca+ca.ab)=$$

$$=2abc(a+b+c)$$

Equality holds for an equilateral triangle.