ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a}{h_b}+\frac{\sum_{cyc} r_a r_b}{\sum_{cyc} r_a^2}\ge\frac{8r}{R}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via AM}-\text{GM},\sum_{cyc}\frac{h_a}{h_b}+\frac{\sum_{cyc} r_a r_b}{\sum_{cyc} r_a^2}\ge 3+\frac{s^2}{(4R+r)^2-2s^2}$$

$$=\frac{3(4R+r)^2-5s^2}{(4R+r)^2-2s^2}\overset{\text{Gerretsen}}{\ge}\frac{3(4R+r)^2-5(4R^2+4Rr+3r^2)}{(4R+r)^2-2(16Rr-5r^2)}\overset{?}{\ge}\frac{8r}{R}$$

$$\Leftrightarrow 7t^3-31t^2+45t-222\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\big((t-2)(7t-3)+5\big)\overset{?}{\ge}0$$

$$\to\text{true}\because t\overset{\text{Euler}}{\ge}2\ \therefore\sum_{cyc}\frac{h_a}{h_b}+\frac{\sum_{cyc} r_a r_b}{\sum_{cyc} r_a^2}\ge\frac{8r}{R}\ \forall\,\Delta\text{ABC},$$

$''=''$ iff Δ ABC is equilateral (QED)