ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\frac{1}{p}\left(\frac{R^2}{r^2}-1\right) \le \sum_{cyc} \frac{AH}{ar_a} \le \frac{1}{p}\left(\frac{2R^2}{r^2}-5\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{AH}{ar_a} = \frac{2R\cos A}{(2R\sin A)p\tan\frac{A}{2}} = \sum_{cyc} \frac{1-\tan^2\frac{A}{2}}{2p\tan^2\frac{A}{2}} = \frac{p}{2}\left(\sum_{cyc}\frac{1}{r_a^2}\right) - \frac{3}{2p}$$

$$= \frac{p}{2}\left(\frac{1}{r^2} - \frac{2(4R+r)}{rp^2}\right) - \frac{3}{2p} \therefore \sum_{cyc} \frac{AH}{ar_a} = \frac{p^2-8Rr-5r^2}{2r^2p} \overset{\text{Gerretsen}}{\le}$$

$$\frac{4R^2-4Rr-2r^2}{2r^2p} \overset{\text{Euler}}{\le} \frac{2R^2-4r^2-r^2}{r^2p} \therefore \sum_{cyc} \frac{AH}{ar_a} \le \frac{1}{p}\left(\frac{2R^2}{r^2}-5\right) \text{ and again,}$$

$$\sum_{cyc} \frac{AH}{ar_a} = \frac{p^2-8Rr-5r^2}{2r^2p} \overset{\text{Walker}}{\ge} \frac{2R^2-2r^2}{2r^2p} \therefore \sum_{cyc} \frac{AH}{ar_a} \ge \frac{1}{p}\left(\frac{R^2}{r^2}-1\right) \text{ and so,}$$

$$\frac{1}{p}\left(\frac{R^2}{r^2}-1\right) \le \sum_{cyc} \frac{AH}{ar_a} \le \frac{1}{p}\left(\frac{2R^2}{r^2}-5\right) \forall \text{ acute } \Delta \text{ ABC},$$

$'' ='' $ iff Δ ABC is equilateral (QED)