ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{r_a^2}{\sqrt{9r^2+3r_a^2}} \leq \frac{9R^2}{8r}$$

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$$\sum_{cyc} \frac{r_a^2}{\sqrt{9r^2+3r_a^2}} = \sum_{cyc}\left(r_a.\frac{r_a}{\sqrt{9r^2+3r_a^2}}\right) \overset{CBS}{\leq} \sqrt{\sum_{cyc} r_a^2}.\sqrt{\frac{1}{3}\sum_{cyc}\frac{3r_a^2+9r^2-9r^2}{9r^2+3r_a^2}}$$

$$= \sqrt{(4R+r)^2-2s^2}.\sqrt{1-\sum_{cyc}\frac{r^2}{3r^2+r_a^2}}$$

$$= \sqrt{(4R+r)^2-2s^2}.\sqrt{1-\sum_{cyc}\frac{r^2(3r^2+r_b^2)(3r^2+r_c^2)}{(3r^2+r_a^2)(3r^2+r_b^2)(3r^2+r_c^2)}}$$

$$= \sqrt{(4R+r)^2-2s^2}.\sqrt{1-\frac{27r^6+6r^4.\sum_{cyc} r_a^2+r^2.\sum_{cyc} r_b^2r_c^2}{27r^6+r^2s^4+9r^4.\sum_{cyc} r_a^2+3r^2.\sum_{cyc} r_b^2r_c^2}} =$$

$$\sqrt{(4R+r)^2-2s^2}.\sqrt{\frac{r^2s^4+3r^4((4R+r)^2-2s^2)+2r^2(s^4-2r(4R+r)s^2)}{27r^6+r^2s^4+9r^4((4R+r)^2-2s^2)+3r^2(s^4-2r(4R+r)s^2)}} \overset{?}{\leq} \frac{9R^2}{8r}$$

$$\overset{\text{squaring and re-arranging}}{\Leftrightarrow} 96r^2s^6+(81R^4-768R^2r^2-896Rr^3-368r^4)s^4 - r(486R^5+486R^4r-4096R^3r^2-6144R^2r^3-2304Rr^4-256r^5)s^2 + r^2(2916R^6+1458R^5r-11559R^4r^2-12288R^3r^3-4608R^2r^4-768Rr^5-48r^6) \underset{(*)}{\overset{?}{\geq}} 0 \text{ and } \because P = 96r^2(s^2-16Rr+5r^2)^3 +$$

$$(81R^4-768R^2r^2+3712Rr^3-1808r^4)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\left(\begin{array}{c}\because 81t^4-768t^2+3712t-1808 \left(t=\frac{R}{r}\right) \\ = (t-2)(81t^3+162t^2-444t+2824)+3840 \overset{\text{Euler}}{\geq} 3840 > 0\end{array}\right)$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow (1053R^5-648R^4r-10240R^3r^2+29440R^2r^3-23296Rr^4+5568r^5)s^2 \underset{(**)}{\overset{?}{\geq}}$$

$$32r\left(\begin{array}{c}278R^6-225R^5r-2860R^4r^2+10816R^3r^3-10980R^2r^4+4182Rr^5- \\ 518r^6\end{array}\right)+$$

$$14R^6-9R^5r+8R^4r^2; \because 1053t^5-648t^4-10240t^3+29440t^2-23296t+5568$$

$$= (t-2)\left((t-2)(1053t^3+3564t^2-196t+14400)+35088\right)+18144$$

$$\overset{\text{Euler}}{\geq} 18144 > 0 \therefore \text{LHS of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$\left(1053R^5-648R^4r-10240R^3r^2+29440R^2r^3-23296Rr^4+5568r^5\right)\begin{pmatrix}16Rr-\\5r^2\end{pmatrix}$$

$$\overset{?}{\geq} \text{RHS}_{(**)} \Leftrightarrow 3969t^6-4212t^5-34544t^4+88064t^3-84288t^2+35872t-$$

$$5632 \overset{?}{\geq} 0 \Leftrightarrow (t-2)\begin{pmatrix}(t-2)\begin{pmatrix}3969t^4+11664t^3-3764t^2+\\26352t+36176\end{pmatrix}+\\75168\end{pmatrix} \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{\text{cyc}} \frac{r_a^2}{\sqrt{9r^2+3r_a^2}} \leq \frac{9R^2}{8r}\ \forall\, \Delta\, ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)