ROMANIAN MATHEMATICAL MAGAZINE

In any acute $\Delta$ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a^2}{\sqrt{9r^2+3h_a^2}}\le\frac{9R}{4}$$

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$$\sum_{cyc}\frac{h_a^2}{\sqrt{9r^2+3h_a^2}}=\sum_{cyc}\left(h_a.\frac{h_a}{\sqrt{9r^2+3h_a^2}}\right)\overset{CBS}{\le}$$

$$\sqrt{\sum_{cyc}h_a^2}.\sqrt{\frac{1}{3}\sum_{cyc}\frac{3h_a^2+9r^2-9r^2}{9r^2+3h_a^2}}\le\sqrt{\sum_{cyc}s(s-a)}.\sqrt{1-\sum_{cyc}\frac{r^2}{3r^2+h_a^2}}$$

$$=s.\sqrt{1-\sum_{cyc}\frac{r^2(3r^2+h_b^2)(3r^2+h_c^2)}{(3r^2+h_a^2)(3r^2+h_b^2)(3r^2+h_c^2)}}$$

$$=s.\sqrt{1-\frac{27r^6+\frac{6r^4}{4R^2}.\sum_{cyc}a^2b^2+\frac{r^4s^2}{R^2}.\sum_{cyc}a^2}{27r^6+\frac{4r^4s^4}{R^2}+\frac{9r^4}{4R^2}.\sum_{cyc}a^2b^2+\frac{3r^4s^2}{R^2}.\sum_{cyc}a^2}}$$

$$=s.\sqrt{\frac{3\sum_{cyc}a^2b^2+8s^2\sum_{cyc}a^2+16s^4}{108R^2r^2+16s^4+9\sum_{cyc}a^2b^2+12s^2\sum_{cyc}a^2}}\overset{?}{\le}\frac{9R}{4}$$

$$\Leftrightarrow s^2.\frac{3\sum_{cyc}a^2b^2+8s^2\sum_{cyc}a^2+16s^4}{108R^2r^2+16s^4+9\sum_{cyc}a^2b^2+12s^2\sum_{cyc}a^2}\overset{?}{\le}\frac{81R^2}{16}$$

$$\Leftrightarrow -560s^6+(3969R^2+1408Rr+160r^2)s^4-$$
$$r(13608R^3+1254R^2r+384Rr^2+48r^3)s^2+$$
$$R^2r^2(20412R^2+5832Rr+729r^2)\underset{(*)}{\overset{?}{\ge}}0;\text{now, Rouche}\Rightarrow s^2-(m-n)\ge 0$$

and $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{①}{\le}0\Rightarrow P=$$
$$-560s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\ge 0\ \therefore\ \text{in order to prove }(*),$$

it suffices to prove : LHS of $(*)\overset{?}{\ge}P\Leftrightarrow(1729R^2-9792Rr+1280r^2)s^4+$
$$r(22232R^3+25626R^2r+6336Rr^2+512r^3)s^2+$$
$$R^2r^2(20412R^2+5832Rr+729r^2)\underset{(**)}{\overset{?}{\ge}}0\text{ and it's trivially true if :}$$

$1729R^2-9792Rr+1280r^2\ge 0$ and when : $1729R^2-9792Rr+1280r^2<0$,

$$\text{then : } Q = (1729R^2 - 9792Rr + 1280r^2)\begin{pmatrix} s^4 - (4R^2 + 20Rr - 2r^2)s^2 + \\ r(4R+r)^3 \end{pmatrix} \overset{\text{via } ①}{\geq} 0$$

$$\therefore \text{ in order to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} Q$$

$$\Leftrightarrow (1729R^4 + 4411R^3r - 42138R^2r^2 + 12880Rr^3 - 512r^4)s^2 \underset{(***)}{\overset{?}{\geq}}$$

$$16r\begin{pmatrix} 1729R^5 - 8815R^4r - 5831R^3r^2 - 861R^2r^3 + \\ 87Rr^4 + 20r^5 \end{pmatrix} + 13R^4r + R^3r^2 + 10R^2r^3$$

**Case 1** $1729R^4 + 4411R^3r - 42138R^2r^2 + 12880Rr^3 - 512r^4 \geq 0$ *and* then :

$$LHS_{(***)} \overset{\text{Gerretsen}}{\geq} (1729R^4 + 4411R^3r - 42138R^2r^2 + 12880Rr^3 - 512r^4)\begin{pmatrix} 16Rr \\ -5r^2 \end{pmatrix}$$

$$\overset{?}{\geq} RHS_{(***)} \Leftrightarrow 16(6342t^4 - 18843t^3 + 13454t^2 - 2312t + 70) + 7t^4 + 4t^3 + 4t^2$$

$$\overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(16(6342t^3 - 6158t^2 + 1138t - 35) + 7t^3 + 2t^2 + 8t) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true}$$

**Case 2** $1729R^4 + 4411R^3r - 42138R^2r^2 + 12880Rr^3 - 512r^4 < 0$ *and* then :

$$LHS_{(***)} \overset{\text{Gerretsen}}{\geq} \begin{pmatrix} 1729R^4 + 4411R^3r - 42138R^2r^2 + \\ 12880Rr^3 - 512r^4 \end{pmatrix}(4R^2 + 4Rr + 3r^2) \overset{?}{\geq}$$

$$RHS_{(***)} \Leftrightarrow 3458t^6 - 1552t^5 - 2347t^4 - 5252t^3 - 31588t^2 + 17600t - 928 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)(3458t^5 + 5364t^4 + 8381t^3 + 11510t^2 - 8568t + 464) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true} \therefore \text{combining both cases}, (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \frac{h_a^2}{\sqrt{9r^2 + 3h_a^2}} \leq \frac{9R}{4} \ \forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral}$$