ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{3\cot^2 A+1}\ge\frac{3}{2}$$

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$$\sum_{cyc}\frac{1}{3\tan^2\frac{A}{2}+1}=\sum_{cyc}\frac{s^2}{3r_a^2+s^2}=$$

$$=\frac{s^2}{(3r_a^2+s^2)(3r_b^2+s^2)(3r_c^2+s^2)}\cdot\sum_{cyc}\left((3r_b^2+s^2)(3r_c^2+s^2)\right)$$

$$=\frac{s^2\left(9\left(\left(\sum_{cyc}r_ar_b\right)^2-2r_ar_br_c\left(\sum_{cyc}r_a\right)\right)+6\left(\left(\sum_{cyc}r_a\right)^2-2\sum_{cyc}r_ar_b\right)+3s^4\right)}{27r^2s^4+9s^2\left(\left(\sum_{cyc}r_ar_b\right)^2-2r_ar_br_c\left(\sum_{cyc}r_a\right)\right)+3s^4\left(\left(\sum_{cyc}r_a\right)^2-2\sum_{cyc}r_ar_b\right)+s^6}$$

$$=\frac{s^2\left(9\left(s^4-2s^2r(4R+r)\right)+6((4R+r)^2-2s^2)+3s^4\right)}{27r^2s^4+9s^2(s^4-2s^2r(4R+r))+3s^4((4R+r)^2-2s^2)+s^6}\overset{?}{\ge}\frac{3}{2}$$

$$\Leftrightarrow 4R^2+8Rr-5r^2-s^2\overset{?}{\ge}0\Leftrightarrow 4R^2+4Rr+3r^2-s^2+4r(R-2r)\overset{?}{\ge}0\to\text{true}$$

$$\because 4R^2+4Rr+3r^2\overset{\text{Gerretsen}}{\ge}s^2\text{ and }R\overset{\text{Euler}}{\ge}2r\therefore\sum_{cyc}\frac{1}{3\tan^2\frac{A}{2}+1}\ge\frac{3}{2}\ \forall\,\Delta\,ABC$$

and implementing it on a triangle with angles : $(\pi-2A),(\pi-2B),(\pi-2C)$,

we get : $\sum_{cyc}\frac{1}{3\tan^2\frac{\pi-2A}{2}+1}\ge\frac{3}{2}\Rightarrow\sum_{cyc}\frac{1}{3\cot^2 A+1}\ge\frac{3}{2}$ **and since the latter**

triangle is an acute one $\therefore\sum_{cyc}\frac{1}{3\cot^2 A+1}\ge\frac{3}{2}\ \forall$ **acute** $\Delta\,ABC$,

$''=''$ **iff Δ ABC is equilateral (QED)**