ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\prod_{cyc}\frac{h_b+h_c}{2h_a}\le\frac{R}{2r}\cdot\frac{\sum_{cyc}h_a^2}{\sum_{cyc}h_ah_b}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}\frac{h_b+h_c}{2h_a}\overset{?}{\le}\frac{R}{2r}\cdot\frac{\sum_{cyc}h_a^2}{\sum_{cyc}h_ah_b}\Leftrightarrow\frac{1}{8}\cdot\prod_{cyc}\frac{ca+ab}{bc}\overset{?}{\le}\frac{R}{2r}\cdot\frac{\sum_{cyc}b^2c^2}{\sum_{cyc}(bc.ca)}$$

$$\Leftrightarrow\frac{1}{8}\cdot\frac{4Rrs.2s(s^2+2Rr+r^2)}{16R^2r^2s^2}\overset{?}{\le}\frac{R}{2r}\cdot\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{8Rrs^2}$$

$$\Leftrightarrow(R-r)s^4-r(8R^2+r^2)s^2+Rr^2(4R+r)^2\underbrace{\overset{?}{\ge}}_{(*)}0$$

Now, LHS of $(*)\overset{\text{Gerretsen}}{\ge}\Big((R-r)(16Rr-5r^2)-r(8R^2+r^2)\Big)s^2+$
$Rr^2(4R+r)^2\overset{?}{\ge}0\Leftrightarrow(8R^2-21Rr+4r^2)s^2+Rr(4R+r)^2\underbrace{\overset{?}{\ge}}_{(**)}0$ and it's trivially true when : $8R^2-21Rr+4r^2\ge0$ and when : $8R^2-21Rr+4r^2<0$, then : LHS of $(**)\overset{\text{Gerretsen}}{\ge}(8R^2-21Rr+4r^2)(4R^2+4Rr+3r^2)+Rr(4R+r)^2$

$$\overset{?}{\ge}0\Leftrightarrow16t^4-18t^3-18t^2-23t+6\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(16t^3+14t^2+10t-3)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\prod_{cyc}\frac{h_b+h_c}{2h_a}\le\frac{R}{2r}\cdot\frac{\sum_{cyc}h_a^2}{\sum_{cyc}h_ah_b}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$