ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$6r \le \sqrt[3]{(h_a+h_b)(h_b+h_c)(h_a+h_c)} \le 3R$$

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$$\sqrt[3]{(h_a+h_b)(h_b+h_c)(h_a+h_c)} \overset{AM-GM}{\ge}$$

$$\ge \sqrt[3]{2\sqrt{h_a h_b}.2\sqrt{h_b h_c}.2\sqrt{h_a h_c}} = 2\sqrt[3]{h_a h_b h_c} =$$

$$= 2\sqrt[3]{\frac{2F^2}{R}} \ge 2\sqrt[3]{\frac{2S^2 r^2}{R}} \ge 2\sqrt[3]{\frac{27Rr.r^2}{R}} = 6r \quad (LHS)$$

$$\sqrt[3]{(h_a+h_b)(h_b+h_c)(h_a+h_c)} \overset{AM-GM}{\le} \frac{2}{3}(h_a+h_b+h_c) \le$$

$$\le \frac{2}{3}(m_a+m_b+m_c) \overset{Gotman}{\le} \frac{2}{3}.\frac{9R}{2} = 3R \ (RHS)$$

$$Equality\ \ holds\ \ for\ \ a=b=c.$$