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In any Δ ABC the following relationship holds :

$$9(R-r) \le \sum_{cyc} \frac{m_a^2}{r_a} \le \frac{9R^3}{8r^2}$$

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$$\sum_{cyc} \frac{m_a^2}{r_a} = \frac{1}{4rs}.\sum_{cyc}\left((s-a)\left(2\sum_{cyc} a^2 - 3a^2\right)\right)$$

$$= \frac{s^2-4Rr-r^2}{rs}.\sum_{cyc}(s-a) - \frac{3}{4rs}.\left(s\sum_{cyc} a^2 - \sum_{cyc} a^3\right)$$

$$= \frac{s^2-4Rr-r^2}{r} - \frac{3}{4rs}.\left(2s(s^2-4Rr-r^2) - 2s(s^2-6Rr-3r^2)\right)$$

$$\stackrel{\boxed{=}}{} \frac{s^2-7Rr-4r^2}{r} \stackrel{\text{Gerretsen}}{\le} \frac{4R^2-3Rr-r^2}{r} \stackrel{?}{\le} \frac{9R^3}{8r^2}$$

$$\Leftrightarrow 9R^3 - 32R^2r + 24Rr^2 + 8r^3 \stackrel{?}{\ge} 0 \Leftrightarrow (R-2r)\left((R-2r)(9R+4r)+4r^2\right) \stackrel{?}{\ge} 0$$

$$\rightarrow \text{true} \because R \stackrel{\text{Euler}}{\ge} 2r \therefore \sum_{cyc} \frac{m_a^2}{r_a} \le \frac{9R^3}{8r^2} \text{ and again, } \sum_{cyc} \frac{m_a^2}{r_a} = \frac{s^2-7Rr-4r^2}{r}$$

$$\stackrel{\text{Gerretsen}}{\ge} \frac{9Rr-9r^2}{r} \therefore \sum_{cyc} \frac{m_a^2}{r_a} \ge 9(R-r) \text{ and so,}$$

$$9(R-r) \le \sum_{cyc} \frac{m_a^2}{r_a} \le \frac{9R^3}{8r^2}\ \forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$