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In any Δ ABC the $following$ $relationship$ holds :

$$\frac{r}{2R^3} \le \sum_{cyc} \frac{\cos A}{a(b+c)} \le \frac{R^3}{128r^5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{\cos A}{a(b+c)} \boxed{=} \frac{1}{2abc.\,2s(s^2+2Rr+r^2)} \cdot \sum_{cyc}\left((b^2+c^2-a^2)\left(a^2+\sum_{cyc} ab\right)\right)$$

$$= \frac{1}{16Rrs^2(s^2+2Rr+r^2)} \cdot \left(\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right) + \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)\right)$$

$$\boxed{=} \frac{8r^2s^2+s^4-(4Rr+r^2)^2}{8Rrs^2(s^2+2Rr+r^2)} \overset{?}{\ge} \frac{r}{2R^3}$$

$$\Leftrightarrow (R^2-4r^2)s^4 + r^2(8R^2-8Rr-4r^2)s^2 - R^2r^2(4R+r)^2 \boxed{\overset{?}{\underset{(*)}{\ge}}} 0$$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\ge} (R^2-4r^2)(16Rr-5r^2)s^2 + r^2(8R^2-8Rr-4r^2)s^2 -$
$R^2r^2(4R+r)^2 \overset{?}{\ge} 0 \Leftrightarrow (16R^3+3R^2r-72Rr^2+16r^3)s^2 \boxed{\overset{?}{\underset{(**)}{\ge}}} R^2r(4R+r)^2$

and $again$, RHS of $(**) \overset{\text{Gerretsen}}{\ge} (16R^3+3R^2r-72Rr^2+16r^3)(16Rr-5r^2)$

$$\left(\begin{array}{c}\because 16R^3+3R^2r-72Rr^2+16r^3 \\ = (R-2r)(16R^2+35Rr-2r^2)+12r^3 \overset{Euler}{\ge} 12r^3 > 0\end{array}\right) \overset{?}{\ge} R^2r(4R+r)^2$$

$$\Leftrightarrow 30R^4 - 5R^3r - 146R^2r^2 + 77Rr^3 - 10r^4 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R-2r)(30R^3+55R^2r-36Rr^2+5r^3) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{\cos A}{a(b+c)} \ge \frac{r}{2R^3} \text{ and again, } \sum_{cyc} \frac{\cos A}{a(b+c)} \overset{?}{\le} \frac{R^3}{128r^5}$$

$$\Leftrightarrow \frac{8r^2s^2+s^4-(4Rr+r^2)^2}{8Rrs^2(s^2+2Rr+r^2)} \overset{?}{\le} \frac{R^3}{128r^5}$$

$$\Leftrightarrow (R^4-16r^4)s^4 + r(2R^5+R^4r-128r^5)s^2 + 16r^6(4R+r)^2 \boxed{\overset{?}{\underset{(\bullet)}{\ge}}} 0 \text{ and now,}$$

LHS of $(\bullet) \overset{\text{Gerretsen}}{\ge} (R^4-16r^4)(16Rr-5r^2)s^2 + r(2R^5+R^4r-128r^5)s^2 +$

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$$16r^6(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow \left(9R^5-2R^4r-128Rr^4-24r^5\right)s^2+8r^5(4R+r)^2 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 0$$

and it's trivi$ally$ true when : $9R^5-2R^4r-128Rr^4-24r^5 \geq 0$ and when :

$$9R^5-2R^4r-128Rr^4-24r^5<0, \text{then : LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq}$$

$$\left(9R^5-2R^4r-128Rr^4-24r^5\right)(4R^2+4Rr+3r^2)+8r^5(4R+r)^2$$

$$\Leftrightarrow 36t^7+28t^6+19t^5-6t^4-512t^3-480t^2-416t-64 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(36t^6+100t^5+219t^4+432t^3+352t^2+224t+32\right) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true } \therefore \sum_{\text{cyc}} \frac{\cos A}{a(b+c)} \leq \frac{R^3}{128r^5} \text{ and so, combining,}$$

$$\frac{r}{2R^3} \leq \sum_{\text{cyc}} \frac{\cos A}{a(b+c)} \leq \frac{R^3}{128r^5} \;\forall\, \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$