ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{2}{r} \le \sum \frac{h_b + h_c}{h_a^2} \le \frac{R}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{h_b + h_c}{h_a^2} = \sum \frac{h_a + h_b + h_c - h_a}{h_a^2} = \sum h_a \cdot \sum \frac{1}{h_a^2} - \sum \frac{1}{h_a} =$$

$$= \frac{bc + ca + ab}{2R} \cdot \frac{a^2 + b^2 + c^2}{4r^2 s^2} - \frac{1}{r} = \frac{(s^2 + r^2 + 4Rr)}{2R} \frac{2(s^2 - r^2 - 4Rr)}{4r^2 s^2} - \frac{1}{r}$$

$$= \frac{s^4 - r^2(4R + r)^2}{4Rr^2 s^2} - \frac{1}{r} = \frac{s^2}{4Rr^2} - \frac{1}{4R}\left(\frac{4R + r}{s}\right)^2 - \frac{1}{r} \overset{\substack{Doucet \\ Gerretsen}}{\le}$$

$$\le \frac{4R^2 + 4Rr + 3r^2}{4Rr^2} - \frac{3}{4R} - \frac{1}{r} = \frac{R}{r^2} + \frac{1}{r} + \frac{3}{4R} - \frac{3}{4R} - \frac{1}{r} = \frac{R}{r^2}$$

$$\sum \frac{1}{h_b + h_c} \overset{AM-GM}{\le} \frac{1}{4} \sum \left(\frac{1}{h_b} + \frac{1}{h_c}\right) = \frac{1}{2} \sum \frac{1}{h_a} = \frac{1}{2r} \ (1)$$

$$\sum \frac{h_b + h_c}{h_a^2} = \sum \frac{\frac{1}{h_a^2}}{\frac{1}{h_b + h_c}} \overset{Bergstrom}{\ge} \frac{\left(\sum \frac{1}{h_a}\right)^2}{\sum \frac{1}{h_b + h_c}} \overset{(1)}{\ge} \frac{\left(\frac{1}{r}\right)^2}{\frac{1}{2r}} = \frac{2}{r}$$

Equality holds for an equilateral triangle.