ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$12r \le \sum \frac{m_a}{\cos^2\frac{A}{2}} \le \frac{3R^2}{r}$$

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$$WLOG\ a \ge b \ge c\ then\ m_a \le m_b \le m_c\ and\ \cos\frac{A}{2} \le \cos\frac{B}{2} \le \cos\frac{C}{2}$$

$$\sum \frac{m_a}{\cos^2\frac{A}{2}} \overset{Chebyshev}{\le} \frac{1}{3}\left(\sum m_a\right)\left(\sum \frac{1}{\cos^2\left(\frac{A}{2}\right)}.\right) = \frac{1}{3}\left(\sum m_a\right)\left(\sum \sec^2\left(\frac{A}{2}\right)..\right) \le$$

$$\overset{Gotman\ II}{\le} \frac{1}{3}\cdot\frac{9R}{2}\left(\frac{s^2+(4R+r)^2}{s^2}\right) \overset{Doucet}{\le} \frac{3R}{2}\left(\frac{\frac{(4R+r)^2}{3}+(4R+r)^2}{3r(4R+r)}\right) =$$

$$= \frac{3R}{2}\cdot\frac{4(4R+r)}{9r} \overset{Euler}{\le} \frac{2R}{3r}\left(\frac{9R}{2}\right) = \frac{3R^2}{r}$$

$$\sum \frac{m_a}{\cos^2\frac{A}{2}} = \sum \frac{\sec^2\frac{A}{2}}{\frac{1}{m_a}}. \overset{Bergstrom}{\ge} \frac{\left(\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\right)^2}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}} \overset{\substack{Jensen \\ m_a \ge h_a}}{\ge}$$

$$\ge \frac{\left(3\sec\frac{\pi}{6}\right)^2}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}} = \frac{\left(3\times\frac{2}{\sqrt{3}}\right)^2}{\frac{1}{r}} = 12r$$

Equality holds for an equilateral triangle.