ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a^2}{h_b+h_c}\le\sum_{cyc}\frac{a^2}{r_b+r_c}$$

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$$\sum_{cyc}\frac{a^2}{h_b+h_c}\overset{?}{\le}\sum_{cyc}\frac{a^2}{r_b+r_c}\Leftrightarrow 2R\sum_{cyc}\frac{a^2}{ca+ab}\overset{?}{\le}\sum_{cyc}\frac{16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}}{4R\cos^2\frac{A}{2}}$$

$$\Leftrightarrow 2R\sum_{cyc}\frac{2s-(b+c)}{b+c}\overset{?}{\le}4R.\frac{2R-r}{2R}$$

$$\Leftrightarrow\frac{2s}{2s(s^2+2Rr+r^2)}.\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)\overset{?}{\le}\frac{2R-r}{R}+3$$

$$\Leftrightarrow\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}\overset{?}{\le}\frac{5R-r}{R}\Leftrightarrow s^2\overset{?}{\le}6R^2+2Rr-r^2$$

$$\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-2(R+r)(R-2r)\overset{?}{\le}0$$

$$\to\text{true}\because s^2\overset{\text{Gerretsen}}{\le}4R^2+4Rr+3r^2\ \text{and}\ R\overset{\text{Euler}}{\ge}2r$$

$$\therefore\sum_{cyc}\frac{a^2}{h_b+h_c}\le\sum_{cyc}\frac{a^2}{r_b+r_c}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$