ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{h_a\cdot\sqrt{\frac{2}{h_b}+\frac{1}{h_c}}}\geq\frac{1}{\sqrt{r}}$$

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Let $x=\frac{3r}{h_a},y=\frac{3r}{h_b},z=\frac{3r}{h_c}$ and then : $\sum_{cyc}\frac{1}{h_a\cdot\sqrt{\frac{2}{h_b}+\frac{1}{h_c}}}=\sum_{cyc}\frac{1}{\frac{3r}{x}\cdot\sqrt{\frac{2y}{3r}+\frac{z}{3r}}}$

$$=\frac{1}{\sqrt{3r}}\cdot\sum_{cyc}\frac{x}{\sqrt{2y+z}}=\frac{1}{\sqrt{3r}}\cdot\sum_{cyc}\frac{x^2}{x.\sqrt{2y+z}}\overset{\text{Bergstrom}}{\geq}\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(x.\sqrt{2y+z}\right)}$$

$$=\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{2xy+zx}\right)}\overset{\text{CBS}}{\geq}\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{\sum_{cyc}x}.\sqrt{3\sum_{cyc}xy}}$$

$$\geq\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{\sum_{cyc}x}.\sqrt{\left(\sum_{cyc}x\right)^2}}=\frac{1}{\sqrt{3r}}\cdot\sqrt{\sum_{cyc}x}=\frac{\sqrt{3}}{\sqrt{3r}}\left(\because\sum_{cyc}x=\sum_{cyc}\frac{3r}{h_a}=3\right)$$

$=\frac{1}{\sqrt{r}}$ and so, $\sum_{cyc}\frac{1}{h_a\cdot\sqrt{\frac{2}{h_b}+\frac{1}{h_c}}}\geq\frac{1}{\sqrt{r}}$ $\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)