ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the $following$ $relationship$ $holds$:

$$36r \le \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \le \frac{4(R^4 - 7r^4)}{r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \overset{Panaitopol}{\le} \sum_{cyc} \left(\frac{Rs}{a} \left(1 + \cot^2 \frac{A}{2} \right) \right)$$

$$= \frac{Rs(s^2 + 4Rr + r^2)}{4Rrs} + \sum_{cyc} \frac{Rs^3 . bc(s - a)^2}{4Rrs . r^2 s^2}$$

$$= \frac{s^2 + 4Rr + r^2}{4r} + \frac{1}{4r^3} (s^2(s^2 + 4Rr + r^2) - 24Rrs^2 + 8Rrs^2)$$

$$= \frac{s^4 - (12Rr - 2r^2)s^2 + r^3(4R + r)}{4r^3} \overset{Gerretsen}{\le} \frac{(4R^2 - 8Rr + 5r^2)s^2 + r^3(4R + r)}{4r^3}$$

$$\overset{Gerretsen}{\le} \frac{(4R^2 - 8Rr + 5r^2)(4R^2 + 4Rr + 3r^2) + r^3(4R + r)}{4r^3} \overset{?}{\le} \frac{4(R^4 - 7r^4)}{r^3}$$

$$\Leftrightarrow 16r(R^3 - 8r^3) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r \therefore \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \le \frac{4(R^4 - 7r^4)}{r^3} \text{ and again,}$$

$$\sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \overset{AM-GM}{\ge} 3.\sqrt[3]{\frac{m_a m_b m_c}{\frac{r^2}{16R^2}}} \overset{Lascu + AM-GM}{\ge} 3.\sqrt[3]{16R^2 . \frac{\sqrt{s(s-a)} . \sqrt{s(s-b)} . \sqrt{s(s-c)}}{r^2}}$$

$$= 3.\sqrt[3]{\frac{16R^2 s^2}{r}} \overset{Gerretsen + Euler}{\ge} 3.\sqrt[3]{\frac{16R^2 . \frac{27Rr}{2}}{r}} = 18R \overset{Euler}{\ge} 36r \therefore \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \ge 36r$$

$$\text{and so, } 36r \le \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \le \frac{4(R^4 - 7r^4)}{r^3} \forall \Delta ABC,$$

$'' ='' $ iff Δ ABC is $equilateral$ (QED)