ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC the following relationship holds :

$$\frac{9}{16} \le \left(\sum_{cyc} \cos A\right)\left(\sum_{cyc} \cos^3 A\right) \le 3 - 39\left(\frac{r}{R}\right)^4$$

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$$\sum_{cyc} \cos^3 A = \sum_{cyc} \cos A - \sum_{cyc} \cos A \sin^2 A$$

$$= \sum_{cyc} \cos A - \sum_{cyc} \sin^2 A + 2\sum_{cyc} \sin^2 \frac{A}{2} \sin^2 A$$

$$= \frac{R+r}{R} - \frac{s^2 - 4Rr - r^2}{2R^2} + \frac{2}{4Rrs.4R^2} . \sum_{cyc}\left(a^3(-s^2 + sa + bc)\right)$$

$$= \frac{R+r}{R} - \frac{s^2 - 4Rr - r^2}{2R^2} +$$

$$\frac{2}{4Rrs.4R^2} . \left(\begin{array}{c}-2s^3(s^2 - 6Rr - 3r^2) + 2s((s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 8r^2s^2) + \\ 8Rrs(s^2 - 4Rr - r^2)\end{array}\right)$$

$$= \frac{4R^3 + 12R^2r + 6Rr^2 + r^3 - 3rs^2}{4R^3} \therefore \left(\sum_{cyc} \cos A\right)\left(\sum_{cyc} \cos^3 A\right)$$

$$= \frac{R+r}{R} . \frac{4R^3 + 12R^2r + 6Rr^2 + r^3 - 3rs^2}{4R^3} \to (1)$$

$$\text{Now, } \frac{R+r}{R} . \frac{4R^3 + 12R^2r + 6Rr^2 + r^3 - 3rs^2}{4R^3} \overset{\text{Gerretsen}}{\le}$$

$$\frac{R+r}{R} . \frac{4R^3 + 12R^2r + 6Rr^2 + r^3 - 3r(16Rr - 5r^2)}{4R^3} \overset{?}{\le} 3 - 39\left(\frac{r}{R}\right)^4$$

$$\Leftrightarrow 4t^4 - 8t^3 + 15t^2 + 13t - 86 \overset{?}{\ge} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow 4t^3(t-2) + 15(t^2 - 4) + 13(t-2) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \left(\sum_{cyc} \cos A\right)\left(\sum_{cyc} \cos^3 A\right) \le 3 - 39\left(\frac{r}{R}\right)^4 \text{ and again, via (1) and Gerretsen,}$$

$$\left(\sum_{cyc} \cos A\right)\left(\sum_{cyc} \cos^3 A\right) \ge$$

$$\frac{R+r}{R} . \frac{4R^3 + 12R^2r + 6Rr^2 + r^3 - 3r(4R^2 + 4Rr + 3r^2)}{4R^3} \overset{?}{\ge} \frac{9}{16}$$

$$\Leftrightarrow 7t^4 + 16t^3 - 24t^2 - 56t - 32 \overset{?}{\ge} 0 \Leftrightarrow (t-2)(7t^3 + 30t^2 + 36t + 16) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \left(\sum_{cyc} \cos A\right)\left(\sum_{cyc} \cos^3 A\right) \geq \frac{9}{16} \text{ and so,}$$

$$\frac{9}{16} \leq \left(\sum_{cyc} \cos A\right)\left(\sum_{cyc} \cos^3 A\right) \leq 3 - 39\left(\frac{r}{R}\right)^4 \ \forall \Delta ABC,$$

$'' ='' $ iff $\Delta$ ABC is equilateral (QED)