ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum (h_b - h_c)(b-c) + 4s(R-2r) \geq 0$$

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$$\sum (h_b - h_c)(b-c) = \frac{1}{2R}\sum (ac-ab)(b-c) =$$

$$= -\frac{1}{2R}\sum a(b-c)^2 = -\frac{1}{2R}\left(\sum a(b^2+c^2) - 6abc\right) =$$

$$= -\frac{1}{2R}\left(\sum a \sum a^2 - \sum a^3 - 6abc\right) =$$

$$= \frac{1}{2R}\left(24Rrs + 2s(s^2-6Rr-3r^2) - 2s.2(s^2-r^2-4Rr)\right)$$

$$= \frac{s}{R}(14Rr - r^2 - s^2) \overset{Gerretsen}{\geq}$$

$$\geq \frac{s}{R}(14Rr - r^2 - 4R^2 - 4Rr - 3r^2) = \frac{s}{R}(10Rr - 4r^2 - 4R^2)$$

$$We\ need\ to\ show:$$

$$\sum (h_b - h_c)(b-c) + 4s(R-2r) \geq 0$$

$$\frac{s}{R}(10Rr - 4r^2 - 4R^2) + 4s(R-2r) \geq 0 \ or\ \frac{s}{R}(2Rr - 4r^2) \geq 0$$

$$\frac{2sr}{R}(R-2r) \geq 0 \ true\ by\ \ Euler.$$

Equality holds for an equilateral triangle.